Identifying Inverse Dominationin the Join of Two Graphs

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Abstract: Let $G = (V(G), E(G))$ be a nontrivial connected simple graph. A subset S of $V(G)$ is a dominating set of G if for every $v \in V(G) \setminus S$, there exists $x \in S$ such that $xv \in E(G)$.Let D be a minimum dominating set of G. If $S \subseteq V(G)\backslash D$ is a dominating set of G, then S is called an inverse dominating set with respect to D. An inverse dominating set $S \subseteq V(G) \setminus D$ is an identifying inverse dominating set of G if for every $v \in V(G)$, $N_G[v] \cap S$ is distinct. The minimum cardinality of an identifying inverse dominating set of G, denoted by $\gamma^{ID(-1)}(G)$, is called the identifying inverse domination number of G. In this paper, we give the identifying inverse dominating set and the identifying inverse domination number in the join of two connected nontrivial graphs.

Keywords: dominating set,identifying code, inverse dominating set, join of two graphs

I. Introduction

A graph G is a pair $(V(G), E(G))$, where $V(G)$ is a finite nonempty set called the vertex-set of G and $E(G)$ is a set of unordered pairs $\{u, v\}$ (or simply uv) of distinct elements from $V(G)$ called the edge-set of G. The elements of $V(G)$ are called vertices and the cardinality $|V(G)|$of $V(G)$ is the order of G. The elements of $E(G)$ are called edges and the cardinality $|E(G)|$ of $E(G)$ is the size of G. If $|V(G)| = 1$, then G is called a trivial graph. If$E(G) = \emptyset$, then G is called an empty graph. The open neighborhood of a vertex $v \in V(G)$ is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The elements of $N_G(v)$ are called neighbors of v. The closed neighborhood of $v \in V(G)$is the set $N_G[v] = N_G(v) \cup \{v\}$. If $X \subseteq V(G)$, the open neighborhood of X in G is the set $N_G(X) = \bigcup_{v \in X} N_G(v)$. The closed neighborhood of X in G is the set $N_G[X] = \bigcup_{v \in X} N_G[v] = N_G(X) \cup X$. When no confusion arises, $N_G[x]$ [res. $N_G(x)$] will be denoted by $N[x]$ [resp. $N(x)$]. Au-v walk in G is a sequence of vertices in G, beginning with u and ending at v such that consecutive vertices in the sequence are adjacent. Au-v walk in a graph in which no vertices are repeated is a u-v path.IfG contains a u-$v$path, then u and v are said to be connected and u is connected to v (and v is connected to u). A graph G is connected if every two vertices of G are connected, that is, if$G$ contains a u-v path for every pair u, v of vertices of G. Since every vertex is connected to itself, the trivial graph is connected. For the general terminology in graph theory, readers may refer to [1].

Domination in graph was introduced by Claude Berge in 1958 and Oystein Ore in 1962 [2]. Following an article [3] by Ernie Cockayne and Stephen Hedetniemi in 1977, the domination in graphs became an area of study by many researchers. A subset S of $V(G)$ is a dominating set of Gif for every $v \in V(G) \setminus S$, there exists $x \in S$ such that $xv \in E(G)$, that is, $N[S] = V(G)$. The domination number $\gamma(G)$ of G is the smallest cardinality of a dominating set of G. Some studies on domination in graphs were found in the papers [4-17].

An identifying code of a graph G is a dominating set $C \subseteq V(G)$ such that for every $v \in V(G)$, $N_G[v] \cap C$ is distinct. The minimum cardinality of an identifying code of G, denoted by $\gamma^{ID}(G)$, is called the identifying code number of G. The identifying code of a graph was studied in 1998 by M.G. Karpovsky, et.al [18] in their paper "On a new class of codes for identifying vertices in graphs". They observed that the concept of identifying codes is that a graph is identifiable if and only if it is twin-free. A vertex x is a twin of another vertex y if $N[x] = N[y]$. A graph G is called twin-free if no vertex has a twin. From a computational point of view, it is shown that given a graph G, finding the exact value of $\gamma^{ID}(G)$ is in the class of NP-hard problems. Some related studies of identifying domination in graphs are found in [19-23].

Let D be a minimum dominating set of G. If $S \subseteq V(G)\backslash D$ is a dominating set of G, then S is called an inverse dominating set with respect to D. The inverse domination number of G is the minimum cardinality of an inverse dominating set of G, denoted by $\gamma^{-1}(G)$. The inverse domination in graph was first found in the paper of Kulli [24]. Some related studies of inverse domination in graphs are found in [25-35].

Motivated by the introduction of the identifying dominating sets and the inverse dominating sets, a new variant of domination in graphs is introduced in this paper. Let $G = (V(G), E(G))$ be a nontrivial connected simple graph. An inverse dominating set $S \subseteq V(G) \setminus D$ is an identifying inverse dominating set of G if for every $v \in V(G)$, $N_G[v] \cap S$ is distinct. The minimum cardinality of an identifying inverse dominating set of G, denoted by $\gamma^{ID(-1)}(G)$, is called the identifying inverse domination number of G. In this paper, wegive the identifying inverse dominating set and the identifying inverse domination number in the join of two connected nontrivial graphs.

II. Results

Definition 2.1 The join of two graphs G and H is the graph $G+H$ with vertex-set $V(G+H) = V(G) \cup V(H)$ and edge-set $E(G+H) = E(G) \cup E(H) \cup \{uv: u \in V(G\}, v \in V(H)\}$.
The following, shows some results onidentifying inverse dominating sets in the join of two nontrivial connected graphs$G$ and H.

Lemma 2.2 Let $G = P_3 = [u_1, u_2, u_3]$and $H = P_n$ where $n \geq 4$. Then $S = S_G \cup S_H \subset V(G+H)$ is an identifying inverse dominating set of $G + H$,if $S_G = \{u_1, u_3\}$ and ($S_H = V(H)$ or $S_H \subset V(H)$ is an identifying code of H with $n \neq 4$).
Proof. Let $G = P_3 = [u_1, u_2, u_3]$and $H = P_n$ where $n \geq 4$.
Suppose that $S_G = \{u_1, u_3\}$ and $S_H = V(H)$. Let $P_n = [v_1, v_2, \ldots, v_n]$, $n \geq 4$. Then $D_G = \{u_2\}$ is theminimum dominating set of $G+H$ and $S = V(G+H) \setminus D_G = \{u_1, u_3\} \cup V(H) = S_G \cup S_H$ where $S_G = \{u_1, u_3\} \subset V(G)$ and $S_H = V(H)$ is an inverse dominating set of $G+H$ with respect to D_G.Now, $S = \{u_1, u_3\} \cup \{v_1, v_2, \ldots, v_n\}$. If $n = 4$, then$S = \{u_1, u_3, v_1, v_2, v_3, v_4\}$. Observe that the$N_{G+H}[u_1] \cap S = \{u_1, v_1, v_2, v_3, v_4\}$,$N_{G+H}[u_2] \cap S = \{u_1, u_3, v_1, v_2, v_3, v_4\}$,$N_{G+H}[u_3] \cap S = \{u_3, v_1, v_2, v_3, v_4\}$, $N_{G+H}[v_1] \cap S = \{u_1, u_3, v_1, v_2\}$,$N_{G+H}[v_2] \cap S = \{u_1, u_3, v_1, v_2, v_3\}$, $N_{G+H}[v_3] \cap S = \{u_1, u_3, v_2, v_3, v_4\}$, and $N_{G+H}[v_4] \cap S = \{u_1, u_3, v_3, v_4\}$. This implies that for every $x \in V(G+H)$, $N_{G+H}[x] \cap S$ is distinct.By definition, S is an identifying inverse dominating set of $G+H$.Similarly, if $n > 4$, then $S = \{u_1, u_3, v_1, v_2, \ldots, v_n\}$ is an identifying inverse dominating set of $G+H$.
Suppose that $S_G = \{u_1, u_3\}$and $S_H \subset V(H)$ is an identifying code of H with$n \neq 4$. Let$S_H = V(H) \setminus \{v\}$. Consider $n = 5$. Then $S_H = \{v_2, v_3, v_4, v_5\}$or $S_H = \{v_1, v_2, v_3, v_4\}$ is an identifying code of H. Theset$D_G = \{u_2\}$ is the minimum dominating set of $G+H$ and $S = V(G+H) \setminus D_G = \{u_1, u_3\} \cup S_H$ where $S_G = \{u_1, u_3\} \subset V(G)$ and $S_H = \{v_2, v_3, v_4, v_5\}$ is an inverse dominating set of $G+H$ with respect to D_G. Now, $S = \{u_1, u_3\} \cup \{v_2, v_3, v_4, v_5\} = \{u_1, u_3, v_2, v_3, v_4, v_5\}$. Observe that the$N_{G+H}[u_1] \cap S = \{u_1, v_2, v_3, v_4, v_5\}$, $N_{G+H}[u_2] \cap S = \{u_1, u_3, v_2, v_3, v_4, v_5\}$, $N_{G+H}[u_3] \cap S = \{u_3, v_2, v_3, v_4, v_5\}$, $N_{G+H}[v_2] \cap S = \{u_1, u_3, v_2, v_3\}$, $N_{G+H}[v_3] \cap S = \{u_1, u_3, v_2, v_3, v_4\}$, $N_{G+H}[v_4] \cap S = \{u_1, u_3, v_3, v_4, v_5\}$, and $N_{G+H}[v_5] \cap S = \{u_1, u_3, v_4, v_5\}$. This implies that for every $x \in V(G+H)$, $N_G[x] \cap S$ is distinct.Thus,S is an identifying inverse dominating set of $G+H$. Similarly, for $S_H = \{v_1, v_2, v_3, v_4\}$, S is an identifying inverse dominating set of $G+H$. If $n > 5$, then S is an identifying inverse dominating set of $G+H$ by similar arguments.

This completes the proof. ■

Lemma 2.3 Let$G = P_m$ where $m \geq 4$ and$H = P_3 = [v_1, v_2, v_3]$. Then$S = S_G \cup S_H \subset V(G+H)$ is an identifying inverse dominating set of $G + H$,if ($S_G = V(G)$or $S_G \subset V(G)$ is an identifying code of G with$m \neq 4$)and $S_H = \{v_1, v_3\}$.
Proof. Let$G = P_m$ where $m \geq 4$ and$H = P_3 = [v_1, v_2, v_3]$.
Suppose that $S_G = V(G)$ and$S_H = \{v_1, v_3\}$.Let $P_m = [u_1, u_2, \ldots, u_m]$, $m \geq 4$. Then $D_H = \{v_2\}$ is the minimum dominating set of $G+H$ and $S = V(G+H) \setminus D_H = V(G) \cup \{v_1, v_3\} = S_G \cup S_H$ where $S_G = V(G)$ and $S_H = \{v_1, v_3\}$ is an inverse dominating set of $G+H$ with respect to D_H. Now, $S = \{u_1, u_2, \ldots, u_m\} \cup \{v_1, v_3\}$. If $m = 4$, then $S = \{u_1, u_2, u_3, u_4, v_1, v_3\}$. Observe that the$N_{G+H}[v_1] \cap S = \{u_1, u_2, u_3, u_4, v_1\}$, $N_{G+H}[v_2] \cap S = \{u_1, u_2, u_3, u_4, v_1, v_3\}$, $N_{G+H}[v_3] \cap S = \{u_1, u_2, u_3, u_4, v_3\}$, $N_{G+H}[u_1] \cap S = \{u_1, u_2, v_1, v_3\}$, $N_{G+H}[u_2] \cap S = \{u_1, u_3, v_1, v_3\}$, $N_{G+H}[u_3] \cap S = \{u_1, u_2, u_3, v_1, v_3\}$, and $N_{G+H}[u_4] \cap S = \{u_3, u_4, v_1, v_3\}$. This implies that for every $y \in V(G+H)$, $N_{G+H}[y] \cap S$ is distinct.By definition, S is an identifying inverse dominating set of $G+H$. Similarly, if $m > 4$, then $S = \{u_1, u_2, \ldots, u_m, v_1, v_3\}$ is an identifying inverse dominating set of $G+H$.
Suppose that $S_G \subset V(G)$ is an identifying code of G with $m \neq 4$ and $S_H = \{v_1, v_3\}$. Let $S_G = V(G) \setminus \{u\}$. Consider $m = 5$. Then $S_G = \{u_2, u_3, u_4, u_5\}$ or $S_G = \{u_1, u_2, u_3, u_4\}$is an identifying code of G. The set $D_H = \{v_2\}$ is the minimum dominating set of $G+H$ and $S = V(G+H) \setminus D_H = S_G \cup \{v_1, v_3\}$ where $S_G = \{u_2, u_3, u_4, u_5\} \subset V(G)$ and $S_H = \{v_1, v_3\}$ is an inverse dominating set of $G+H$ with respect to D_H.Now, $S = \{u_2, u_3, u_4, u_5\} \cup \{v_1, v_3\} = \{u_2, u_3, u_4, u_5, v_1, v_3\}$. Observe that the $N_{G+H}[u_2] \cap S = \{u_2, u_3, v_1, v_3\}$, $N_{G+H}[u_3] \cap S = \{u_2, u_3, u_4, v_1, v_3\}$, $N_{G+H}[u_4] \cap S = \{u_3, u_4, u_5, v_1, v_3\}$, $N_{G+H}[u_5] \cap S = \{u_4, u_5, v_1, v_3\}$, $N_{G+H}[v_1] \cap S = \{v_1, u_2, u_3, u_4, u_5\}$,$N_{G+H}[v_2] \cap S = \{v_1, v_2, v_3, u_2, u_3, u_4, u_5\}$, and $N_{G+H}[v_3] \cap S =$

$\{v_3, u_2, u_3, u_4, u_5\}$. This implies thatfor every $x \in V(G+H)$, $N_G[x] \cap S$ is distinct.Thus,S is an identifying inverse dominating set of $G+H$.Similarly, for $S_G = \{u_1, u_2, u_3, u_4\}$, S is an identifying inverse dominating set of $G+H$. If $m > 5$, then S is an identifying inverse dominating set of $G+H$ by similar arguments.

This completes the proof. ■

Lemma 2.4 Let$G = P_5 = [u_1, u_2, u_3, u_4, u_5]andH = P_n$,$n \geq 4$. Then $S = S_G \cup S_H \subset V(G+H)$ is an identifying inverse dominating set of $G + H$,if $S_G = \{u_1, u_3, u_5\}$ and S_H is an identifying code of H.
*Proof.*Let$G = P_5 = [u_1, u_2, u_3, u_4, u_5]$ and$H = P_n$, $n \geq 4$. Suppose that $S_G = \{u_1, u_3, u_5\}$ and S_H is an identifying code of H. The set $D_G = \{u_2, u_4\}$ is a minimum dominating set of$G+H$ and $S = V(G+H) \setminus D_G$ is an inverse dominating set of $G+H$ with respect to D_G.

Consider that $n = 4$ andlet $H = P_4 = [v_1, v_2, v_3, v_4]$.Then $S_H = V(H)$or $S_H = V(H) \setminus \{v\}$(where $v = v_1$ or $v = v_4$). If $S_H = V(H)$, then $S = V(G+H) \setminus D_G = \{u_1, u_3, u_5, v_1, v_2, v_3, v_4\} = \{u_1, u_3, u_5\} \cup \{v_1, v_2, v_3, v_4\} = S_G \cup S_H$. Since $N_{G+H}[u_1] \cap S = \{u_1\} \cup V(H)$, $N_{G+H}[u_2] \cap S = \{u_1, u_3\} \cup V(H)$, $N_{G+H}[u_3] \cap S = \{u_3\} \cup V(H)$,$N_{G+H}[u_4] \cap S = \{u_3, u_5\} \cup V(H)$,$N_{G+H}[u_5] \cap S = \{u_5\} \cup V(H)$,
$N_{G+H}[v_1] \cap S = S_G \cup \{v_1, v_2\}$,$N_{G+H}[v_2] \cap S = S_G \cup \{v_1, v_2, v_3\}$, $N_{G+H}[v_3] \cap S = S_G \cup \{v_2, v_3, v_4\}$, and $N_{G+H}[v_4] \cap S = S_G \cup \{v_3, v_4\}$, it follows that for every $x \in V(G+H)$,$N_G[x] \cap S$ is distinct. Thus, $S = S_G \cup S_H \subset V(G+H)$is an identifying inverse dominating set of $G+H$. Similarly, for $S_H = V(H) \setminus \{v_1\}$(or $S_H = V(H) \setminus \{v_2\}$), $S = S_G \cup S_H \subset V(G+H)$ is an identifying inverse dominating set of $G+H$.

Consider that $n \neq 4$ and let $P_n = [v_1, v_2, \ldots, v_n]$ for all $n > 4$.Then $S_H = V(H)$ or $S_H \subset V(H)$.
***Case* 1.**If $S_H = V(H)$, then $S = V(G+H) \setminus D_G = \{u_1, u_3, u_5\} \cup V(H) = S_G \cup S_H$. Let$x, y \in V(G+H)$. Suppose that$x, y \in V(G)$.If $x, y \in S$, then $N_{G+H}[x] \cap S = \{x\} \cup V(H) \neq \{y\} \cup V(H) = N_{G+H}[y] \cap S$. If $x, y \notin S$, say $x = u_2$ and $y = u_4$, then $N_{G+H}[x] \cap S = \{u_1, u_3\} \cup V(H) \neq \{u_3, u_5\} \cup V(H) = N_{G+H}[y] \cap S$.Suppose that $x, y \in V(H)$. Then $x, y \in S$ and $N_{G+H}[x] \cap S = S_G \cup N_H[x] \neq S_G \cup N_H[y] = N_{G+H}[y] \cap S$. Suppose that $x \in V(G)$ and$y \in V(H)$. If $x \in S$, then $N_{G+H}[x] \cap S = \{x\} \cup V(H) \neq S_G \cup N_H[y] = N_{G+H}[y] \cap S$.If $x \notin S$, say $x = u_2$, then $N_{G+H}[x] \cap S = \{u_1, u_3\} \cup V(H) \neq S_G \cup N_H[y] = N_{G+H}[y] \cap S$. Thus, for every $x \in V(G+H)$, $N_G[x] \cap S$ is distinct, that is, S is an identifying code of $G+H$. Since $S = V(G+H) \setminus D_G$ is an inverse dominating set of $G+H$ with respect to D_G, it follows that $S = S_G \cup S_H \subset V(G+H)$ is an identifying inverse dominating set of $G+H$.

***Case* 2.**If $S_H \subset V(H)$, then $S = V(G+H) \setminus D_G = \{u_1, u_3, u_5\} \cup S_H = S_G \cup S_H$. Let $x, y \in V(G+H)$. Suppose that $x, y \in V(G)$. If $x, y \in S$, then $N_{G+H}[x] \cap S = \{x\} \cup S_H \neq \{y\} \cup S_H = N_{G+H}[y] \cap S$. If $x, y \notin S$, say $x = u_2$ and $y = u_4$,then $N_{G+H}[x] \cap S = \{u_1, u_3\} \cup S_H \neq \{u_3, u_5\} \cup S_H = N_{G+H}[y] \cap S$.If $x \in S$ and $y \notin S$, say $y = u_4$, then $N_{G+H}[x] \cap S = \{x\} \cup S_H \neq \{u_3, u_5\} \cup S_H = N_{G+H}[y] \cap S$.Suppose that $x, y \in V(H)$. If $x, y \in S$, then $N_{G+H}[x] \cap S = S_G \cup (N_H[x] \cap S_H) \neq S_G \cup (N_H[y] \cap S_H) = N_{G+H}[y] \cap S$. If $x, y \notin S$, then $N_{G+H}[x] \cap S = S_G \cup (N_H(x) \cap S_H) \neq S_G \cup (N_H(y) \cap S_H) = N_{G+H}[y] \cap S$. If $x \in S$and $y \notin S$, then $N_{G+H}[x] \cap S = S_G \cup (N_H[x] \cap S_H) \neq S_G \cup (N_H(y) \cap S_H) = N_{G+H}[y] \cap S$.Suppose that $x \in V(G)$ and$y \in V(H)$. If $x, y \in S$, then $N_{G+H}[x] \cap S = \{x\} \cup S_H \neq S_G \cup (N_H[y] \cap S_H = N_{G+H}[y] \cap S$. If $x, y \notin S$, say $x = u_2$, then $N_{G+H}[x] \cap S = \{u_1, u_3\} \cup S_H \neq S_G \cup (N_H[y] \cap S_H) = N_{G+H}[y] \cap S$. Thus, for every $x \in V(G+H)$,$N_G[x] \cap S$ is distinct, that is, S is an identifying code of $G+H$. If $x \in S$ and $y \notin S$, then $N_{G+H}[x] \cap S = \{x\} \cup S_H \neq S_G \cup (N_H[y] \cap S_H) = N_{G+H}[y] \cap S$. If $x \notin S$, say $x = u_2$, and $y \in S$, then $N_{G+H}[x] \cap S = \{u_1, u_3\} \cup S_H \neq S_G \cup (N_H[y] \cap S_H) = N_{G+H}[y] \cap S$. Thus, for every $x \in V(G+H)$,$N_G[x] \cap S$ is distinct, that is, S is an identifying code of $G+H$. Since $S = V(G+H) \setminus D_G$ is an inverse dominating set of $G+H$ with respect to D_G, it follows that $S = S_G \cup S_H \subset V(G+H)$ is an identifying inverse dominating set of $G+H$.

This completes the proof. ■

Lemma 2.5 Let$G = P_m$, $m \geq 4$ and $H = P_5 = [v_1, v_2, v_3, v_4, v_5]$. Then $S = S_G \cup S_H \subset V(G+H)$ is an identifying inverse dominating set of $G + H$,if S_G is an identifying code of G and $S_H = \{v_1, v_3, v_5\}$.
*Proof.*Let$G = P_m$, $m \geq 4$ and $H = P_5 = [v_1, v_2, v_3, v_4, v_5]$. Suppose that S_G is an identifying code of G and $S_H = \{v_1, v_3, v_5\}$. The set $D_H = \{v_2, v_4\}$ is a minimum dominating set of $G+H$ and $S = V(G+H) \setminus D_H$ is an inverse dominating set of $G+H$ with respect to D_H.
Consider that $m = 4$ and let $G = P_4 = [u_1, u_2, u_3, u_4]$. Then $S_G = V(G)$ or $S_G = V(G) \setminus \{u\}$ (where $u = u_1$ or $u = u_4$). If $S_G = V(G)$, then $S = V(G+H) \setminus D_H = \{u_1, u_2, u_3, u_4, v_1, v_3, v_5\} = \{u_1, u_2, u_3, u_4\} \cup \{v_1, v_3, v_5\} =$

$S_G \cup S_H$. Since $N_{G+H}[u_1] \cap S = \{u_1, u_2\} \cup S_H$, $N_{G+H}[u_2] \cap S = \{u_1, u_2, u_3\} \cup S_H$,$N_{G+H}[u_3] \cap S = \{u_2, u_3, u_4\} \cup S_H$,$N_{G+H}[u_4] \cap S = \{u_3, u_4\} \cup S_H$,$N_{G+H}[v_1] \cap S = V(G) \cup \{v_1\}$, $N_{G+H}[v_2] \cap S = V(G) \cup \{v_1, v_3\}$, $N_{G+H}[v_3] \cap S = V(G) \cup \{v_3\}$, $N_{G+H}[v_4] \cap S = V(G) \cup \{v_3, v_5\}$, and $N_{G+H}[v_5] \cap S = V(G) \cup \{v_5\}$, it follows that for every $x \in V(G+H)$, $N_G[x] \cap S$ is distinct. Thus, $S = S_G \cup S_H \subset V(G+H)$ is an identifying inverse dominating set of $G+H$. Similarly, for $S_G = V(G) \setminus \{u_1\}$ (or $S_G = V(G) \setminus \{u_2\}$), $S = S_G \cup S_H \subset V(G+H)$ is an identifying inverse dominating set of $G+H$.

Consider that $m \neq 4$ and let $P_m = [u_1, u_2, \dots, u_m]$ for all $m > 4$. Then $S_G = V(G)$ or $S_G \subset V(G)$.

***Case* 1.** If $S_G = V(G)$, then $S = V(G+H) \setminus D_H = V(G) \cup S_H = S_G \cup S_H$. Let $x, y \in V(G+H)$. Suppose that $x, y \in V(G)$. Then $x, y \in S$and $N_{G+H}[x] \cap S = N_G[x] \cup S_H \neq N_G[y] \cup S_H = N_{G+H}[y] \cap S$. Suppose that $x, y \in V(H)$. If $x, y \in S$, then $N_{G+H}[x] \cap S = V(G) \cup \{x\} \neq V(G) \cup \{y\} = N_{G+H}[y] \cap S$. If $x, y \notin S$, say $x = v_2$ and $y = v_4$, then $N_{G+H}[x] \cap S = V(G) \cup \{v_1, v_3\} \neq V(G) \cup \{v_3, v_5\} = N_{G+H}[y] \cap S$. If $x \in S$ and $y \notin S$, say $y = v_2$, then $N_{G+H}[x] \cap S = V(G) \cup \{x\} \neq V(G) \cup \{v_1, v_3\} = N_{G+H}[y] \cap S$.Suppose that $x \in V(G)$ and $y \in V(H)$. If $x, y \in S$, then $N_{G+H}[x] \cap S = N_G[x] \cup S_H \neq V(G) \cup \{y\} = N_{G+H}[y] \cap S$. If $x \in S$ and $y \notin S$, then $N_{G+H}[x] \cap S = N_G[x] \cup S_H \neq V(G) \cup (N_H[y] \cap S_H) = N_{G+H}[y] \cap S$. This implies that for every $x \in V(G+H)$, $N_G[x] \cap S$ is distinct. Thus, $S = S_G \cup S_H \subset V(G+H)$ is an identifying inverse dominating set of $G+H$.

***Case* 2.**If $S_G \subset V(G)$, then $S = V(G+H) \setminus D_H = S_G \cup \{v_1, v_3, v_5\} = S_G \cup S_H$. Let $x, y \in V(G+H)$. Suppose that $x, y \in V(G)$. Then $N_{G+H}[x] \cap S = (N_G[x] \cap S_G) \cup S_H \neq (N_G[y] \cap S_G) \cup S_H = N_{G+H}[y] \cap S$. Suppose that $x, y \in V(H)$.Then $N_{G+H}[x] \cap S = S_G \cup (N_H[x] \cap S_H) \neq S_G \cup (N_H[y] \cap S_H) = N_{G+H}[y] \cap S$. Suppose that $x \in V(G)$ and $y \in V(H)$. Then $N_{G+H}[x] \cap S = (N_G[x] \cap S_G) \cup S_H \neq S_G \cup (N_H[y] \cap S_H) = N_{G+H}[y] \cap S$. Thus, for every $x \in V(G+H)$, $N_G[x] \cap S$ is distinct, that is, S is an identifying code of $G+H$. Since $S = V(G+H) \setminus D_H$ is an inverse dominating set of $G+H$ with respect to D_H, it follows that $S = S_G \cup S_H \subset V(G+H)$ is an identifying inverse dominating set of $G+H$.

This completes the proof. ■

Lemma 2.6 Let $G = P_m = [u_1, u_2, \dots, u_m]$with $m \geq 4$and $H = P_n = [v_1, v_2, \dots, v_n]$ with $n \geq 4$.Then $S = S_G \cup S_H \subset V(G+H)$ is an identifying inverse dominating set of$G+H$,if $S_G \subset V(G)$ is an identifying code of $G$and $S_H \subset V(H)$ is an identifying code of H.
Proof. Let $G = P_m = [u_1, u_2, \dots, u_m]$with $n \geq 4$and $H = P_n = [v_1, v_2, \dots, v_n]$ with $n \geq 4$.If S_G is an identifying code of $G$and $S_H \subset V(H)$ is an identifying code of H, then consider the following.

Case 1. Suppose that$m = 4$, that is, $G = P_4 = [u_1, u_2, u_3, u_4]$.Let$S_G = V(G) \setminus \{u\}$, with $u = u_1$ or $u = u_4$. If $S_G = V(G) \setminus \{u_1\} = \{u_2, u_3, u_4\}$, then for every $u \in V(G)$, the $N_G[u] \cap S_G$ is distinct.Similarly, if $S_G = V(G) \setminus \{u_4\} = \{u_1, u_2, u_3\}$, then for every $u \in V(G)$, the $N_G[u] \cap S_G$ is distinct.For $u \in \{u_1, u_4\} \subset V(G)$and for any $v \in V(H)$with $n \geq 5$, the set $D = \{u, v\}$ is a minimum dominating set of $G+H$. Further, the set $S \subseteq V(G+H) \setminus D$ is an inverse dominating set of $G+H$. Now, to show that S is an identifying code of $G+H$. Let $x, y \in V(G+H)$. Consider the following.

Subcase 1. If $x, y \in V(G)$, then $N_G[x] \cap S_G \neq N_G[y] \cap S_G$ since S_G is an identifying code of G.Hence $N_{G+H}[x] \cap S = N_{G+H}[x] \cap (S_G \cup S_H) \neq N_{G+H}[y] \cap (S_G \cup S_H) = N_{G+H}[y] \cap S$, that is, $N_{G+H}[x] \cap S \neq N_{G+H}[y] \cap S$.

Subcase 2. If $x, y \in V(H)$, then $N_H[x] \cap S_H \neq N_H[y] \cap S_H$ since S_H is an identifying code of H and hence $N_{G+H}[x] \cap S \neq N_{G+H}[y] \cap S$using similar arguments.

Subcase 3. If $x \in V(G)$ and $y \in V(H)$, then $N_{G+H}[x] \cap S = N_{G+H}[x] \cap (S_G \cup S_H) = (N_{G+H}[x] \cap S_G) \cup (N_{G+H}[x] \cap S_H) = (N_G[x] \cap S_G) \cup S_H \neq S_G \cup (N_H[y] \cap S_H) =$
$(N_{G+H}[y] \cap S_G) \cup (N_{G+H}[y] \cap S_H)$=$N_{G+H}[y] \cap (S_G \cup S_H) = N_{G+H}[y] \cap S$, that is, $N_{G+H}[x] \cap S \neq N_{G+H}[y] \cap S$.

In any case, $N_{G+H}[x] \cap S \neq N_{G+H}[y] \cap S$, that is, S is an identifying code of $G+H$. Since S is an inverse dominating set of $G+H$, it follows that$S$ is an identifying inverse dominating set of $G+H$.

Case 2. Suppose that $n = 4$, that is, $H = P_4 = [v_1, v_2, v_3, u_4]$. Let $S_H = V(H) \setminus \{v\}$, with $v = v_1$ or $v = v_4$. If $S_H = V(H) \setminus \{v_1\} = \{v_2, v_3, v_4\}$, then for every $v \in V(H)$, the $N_H[v] \cap S_H$ is distinct. Similarly, if $S_H = V(H) \setminus \{v_4\} = \{v_1, v_2, v_3\}$, then for every $v \in V(H)$, the $N_H[v] \cap S_H$ is distinct. For any $u \in V(G)$with $m \geq$

5and $v \in \{v_1, v_4\} \subset V(H)$, the set $D = \{u, v\}$ is a minimum dominating set of $G + H$. Further, the set $S \subseteq V(G + H) \setminus D$ is an inverse dominating set of $G + H$. Now, to show that S is an identifying code of $G + H$,by similar arguments in Case 1, S is an identifying inverse dominating set of $G + H$.

Case 3. Suppose that $m > 4$ and $n > 4$.

Subcase 1. Suppose that m and n are odd integers. Let $S_G \subseteq V(G) \setminus \{u_r\}$ and $S_H \subseteq V(H) \setminus \{v_s\}$ such that r and s are even integers. The set $D = \{u_r, v_s\}$ is a minimum dominating set of $G + H$ and the set $S \subseteq V(G + H) \setminus D$ is an inverse dominating set of $G + H$ with respect to D.

First, consider that $S_G = V(G) \setminus \{u_r\}$ and $S_H = V(H) \setminus \{v_s\}$. Using the same arguments above, $S = S_G \cup S_H \subset V(G + H)$ is an identifying inverse dominating set of $G + H$.

Next, consider that $S_G \subset V(G) \setminus \{u_r\}$ and $S_H \subset V(H) \setminus \{v_s\}$, say$S_G = \{u_{2i-1}: i = 1,2,3, \dots, \frac{m+1}{2}\}$ and $S_H = \{u_{2i-1}: i = 1,2,3, \dots, \frac{n+1}{2}\}$. Let $x, y \in V(G + H)$. If $x, y \in V(G)$, then $N_G[x] \cap S_G \neq N_G[y] \cap S_G$ because S_G is an identifying code of G. Thus,

$$
\begin{aligned}
&N_{G+H}[x] \cap S = (N_G[x] \cup V(H)) \cap S \\
&= (N_G[x] \cap S) \cup (V(H) \cap S) \\
&= \big(N_G[x] \cap (S_G \cup S_H)\big) \cup \big(V(H) \cap (S_G \cup S_H)\big) \\
&= [(N_G[x] \cap S_G) \cup (N_G[x] \cap S_H)] \cup [(V(H) \cap S_G) \cup (V(H) \cap S_H)] \\
&= [(N_G[x] \cap S_G) \cup \emptyset] \cup [\emptyset \cup S_H] \\
&= (N_G[x] \cap S_G) \cup S_H \\
&= (N_G[x] \cup S_H) \cap (S_G \cup S_H) \\
&\neq (N_G[y] \cup S_G) \cap (S_G \cup S_H) \\
&= (N_G[y] \cap S_G) \cup S_H \\
&= [(N_G[y] \cap S_G) \cup \emptyset] \cup [\emptyset \cup S_H] \\
&= [(N_G[y] \cap S_G) \cup (N_G[y] \cap S_H)] \cup [(V(H) \cap S_G) \cup (V(H) \cap S_H)] \\
&= \big(N_G[y] \cap (S_G \cup S_H)\big) \cup \big(V(H) \cap (S_G \cup S_H)\big) \\
&= (N_G[y] \cap S) \cup (V(H) \cap S) \\
&= (N_G[y] \cup V(H)) \cap S \\
&= N_{G+H}[y] \cap S.
\end{aligned}
$$

That is, $N_{G+H}[x] \cap S \neq N_{G+H}[y] \cap S$, hence S is an identifying code of $G + H$. Since S is an inverse dominating set of $G + H$, it follows that $S = S_G \cup S_H \subset V(G + H)$ is an identifying inverse dominating set of $G + H$.

Finally, consider that $S_G \subset V(G) \setminus \{u_r\}$ and $S_H \subset V(H) \setminus \{v_s\}$, such thatif$x, y, z, w \in V(G)$ and $xy, yz, zw \in E(G)$, then at least $x, y, z \in S_G$. Similarly, if $x, y, z, w \in V(H)$ and $xy, yz, zw \in E(H)$, then at least $x, y, z \in S_H$. The set$S = S_G \cup S_H \subset V(G + H)$ is an identifying inverse dominating set of $G + H$.

Subcase 2. Suppose that m and n are even integers. Let $S_G \subseteq V(G) \setminus \{u\}$ and $S_H \subseteq V(H) \setminus \{v\}$ such that $u \in V(G)$ and $v \in V(H)$. The set $D = \{u, v\}$ is a minimum dominating set of $G + H$ and the set $S \subseteq V(G + H) \setminus D$ is an inverse dominating set of $G + H$ with respect to D.

First, consider that $S_G = V(G) \setminus \{u\}$ and $S_H = V(H) \setminus \{v\}$. Using the same arguments above, $S = S_G \cup S_H \subset V(G + H)$ is an identifying inverse dominating set of $G + H$.

Finally, consider that $S_G \subset V(G) \setminus \{u\}$ and $S_H \subset V(H) \setminus \{v\}$, such that if $x, y, z, w \in V(G)$ and $xy, yz, zw \in E(G)$, then at least $x, y, z \in S_G$. Similarly, if $x, y, z, w \in V(H)$ and $xy, yz, zw \in E(H)$, then at least $x, y, z \in S_H$. The set $S = S_G \cup S_H \subset V(G + H)$ is an identifying inverse dominating set of $G + H$.

Subcase 3. Suppose that m is an even integer and n is an odd integer. By similar arguments Subcases 1 and 2, the set $S = S_G \cup S_H \subset V(G + H)$ is an identifying inverse dominating set of $G + H$.

This completes the proofs. ■

Theorem 2.7 Let $G = P_m = [u_1, u_2, \dots, u_m]$ and $H = P_n = [v_1, v_2, \dots, v_n]$ with $m, n \geq 3$. Then $S = S_G \cup S_H \subset V(G + H)$ is an identifying inverse dominating set of$G + H$,if $S_G \subseteq V(G)$ is an identifying code of $G$and $S_H \subseteq V(H)$ is an identifying code of $H$and one of the following statements holds:
i)$S_G = \{u_1, u_3\}$ and $S_H \subseteq V(H) = \{v_1, v_2, \dots, v_n\}$ with $n \geq 4$or $S_G \subseteq V(G) = \{u_1, u_2, \dots, u_m\}$with $m \geq 4$ and $S_H = \{v_1, v_3\}$.
ii)$S_G = \{u_1, u_3, u_5\}$ and $S_H \subseteq V(H) = \{v_1, v_2, \dots, v_n\}$with $n \geq 4$ or $S_G \subseteq V(G) = \{u_1, u_2, \dots, u_m\}$with $m \geq 4$ and $S_H = \{v_1, v_3, v_5\}$.
iii) $S_G \subseteq V(G) \setminus \{u\}$and $S_H \subseteq V(H) \setminus \{v\}$ with $m \geq 4$ and $n \geq 4$.

*Proof.*Let $G = P_m = [u_1, u_2, \dots, u_m]$ and $H = P_n = [v_1, v_2, \dots, v_n]$ with $m, n \geq 3$. If $S_G \subseteq V(G)$ is an identifying code of G and $S_H \subseteq V(H)$ is an identifying code of H. Consider the following cases.

Case 1. Suppose that statementi) holds. First, consider that $S_G = \{u_1, u_3\}$ and $S_H \subseteq V(H) = \{v_1, v_2, \dots, v_n\}$ with $n \geq 4$. Let $G = P_3 = [u_1, u_2, u_3]$and $H = P_n$ with $n \geq 4$.Then S_G is an identifying code of $G$and S_H is an identifying code of H.By Lemma 2.2, $S = S_G \cup S_H \subset V(G + H)$ is an identifying inverse dominating set of$G + H$.Second, consider that $S_G \subseteq V(G) = \{u_1, u_2, \dots, u_m\}$ with $m \geq 4$ and $S_H = \{v_1, v_3\}$.Let$G = P_m$ where $m \geq 4$ and$H = P_3 = [v_1, v_2, v_3]$.Then S_G is an identifying code of G and S_H is an identifying code of H. By Lemma 2.3, $S = S_G \cup S_H \subset V(G + H)$ is an identifying inverse dominating set of$G + H$.

Case 2. Suppose that statement ii) holds. First, consider that $S_G = \{u_1, u_3, u_5\}$ and $S_H \subseteq V(H) = \{v_1, v_2, \dots, v_n\}$ with $n \geq 4$. Let$G = P_5 = [u_1, u_2, u_3, u_4, u_5]$ and$H = P_n = [v_1, v_2, \dots, v_n]$, $n \geq 4$. Then S_G is an identifying code of G and S_H is an identifying code of H. By Lemma 2.4, $S = S_G \cup S_H \subset V(G + H)$ is an identifying inverse dominating set of$G + H$. Second, consider that $S_G \subseteq V(G) = \{u_1, u_2, \dots, u_m\}$ with $m \geq 4$ and $S_H = \{v_1, v_3, v_5\}$.Let$G = P_m = [u_1, u_2, \dots, u_n]$, $m \geq 4$ and $H = P_5 = [v_1, v_2, v_3, v_4, v_5]$. Then S_G is an identifying code of G and S_H is an identifying code of H. By Lemma 2.5, $S = S_G \cup S_H \subset V(G + H)$ is an identifying inverse dominating set of$G + H$.

Case 3. Suppose that statement iii) holds. Then $S_G \subseteq V(G) \setminus \{u\}$ and $S_H \subseteq V(H) \setminus \{v\}$ with $m \geq 4$ and $n \geq 4$. Let $G = P_m = [u_1, u_2, \dots, u_m]$with $n \geq 4$and $H = P_n = [v_1, v_2, \dots, v_n]$ with $n \geq 4$.Then $S_G \subset V(G)$ is an identifying code of G and $S_H \subset V(H)$ is an identifying code of H. By Lemma 2.6, $S = S_G \cup S_H \subset V(G + H)$ is an identifying inverse dominating set of$G + H$.

This completes the proof. ∎

The following result is a quick consequence of Theorem 2.7.

Corollary 2.8 Let $G = P_m = [u_1, u_2, \dots, u_m]$with odd integer$m \geq 5$and $H = P_n = [v_1, v_2, \dots, v_n]$ with odd integer $n \geq 5$.Then $\gamma^{ID(-1)}(G + H) = \frac{m+n+2}{2}$.
Proof. Let $S_G \subseteq V(G) \setminus \{u_r\}$ and $S_H \subseteq V(H) \setminus \{v_s\}$ such that r and s are even integers. The set $D = \{u_r, v_s\}$ is a minimum dominating set of $G + H$ and the set $S \subseteq V(G + H) \setminus D$ is an inverse dominating set of $G + H$ with respect to D. Consider that $S_G \subset V(G) \setminus \{u_r\}$ and $S_H \subset V(H) \setminus \{v_s\}$. Let$S_G = \{u_{2i-1}: i = 1,2,3, \dots, \frac{m+1}{2}\}$ and $S_H = \{u_{2i-1}: i = 1,2,3, \dots, \frac{n+1}{2}\}$. Then S_G and S_H are identifying code of G and H respectively. Let $x, y \in V(G + H)$. Then $N_{G+H}[x] \cap S \neq N_{G+H}[y] \cap S$(see the proof in Theorem 2.7). Thus,S is an identifying inverse dominating set of $G + H$. Further, $|S| = |S_G \cup S_H| = |S_G| + |S_H| = \frac{m+1}{2} + \frac{n+1}{2} = \frac{m+n+2}{2}$. This implies that $|S| = \frac{m+n+2}{2} \geq \gamma^{ID(-1)}(G + H)$. Suppose that S is not a minimum identifying inverse dominating set of $G + H$. Then there exists $x \in V(G + H)$ such that $S \setminus \{x\}$ is an identifying inverse dominating set of $G + H$. If $x \in V(G)$, then $x \in S_G$. Now, $S_G \setminus \{x\}$ is not a dominating set of G for any $x \in S_G$ contrary to the fact that S_G is an identifying code of G. If $x \in V(H)$, then $x \in S_H$. Now, $S_H \setminus \{x\}$ is not a dominating set of H for any $x \in S_H$ contrary to the fact that S_H is an identifying code of H. This implies that S must be a minimum identifying inverse dominating set of $G + H$. Thus,$|S| = \frac{m+n+2}{2}$, that is,$\gamma^{ID(-1)}(G + H) = \frac{m+n+2}{2}$. ∎

III. Conclusion

In this work, we introduced a new parameter of domination in graphs - the identifying inverse domination in graphs. Some resultson the identifying inverse dominating set in the join of two connected graphs were proven. The identifying inverse domination number inthe join of two graphs were computed. This study will pave a way to new researches such bounds and other binary operations of two graphs. Other parameters involving identifying inverse domination in graphs may also be explored. Finally, the characterization of an identifying inverse domination in graphs and its bounds is a promising extension of this study.

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