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Outer-Connected Fair Domination in the Join of Two Graphs

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Abstract: Let $G = (V(G), E(G))$ be a nontrivial connected simple graph. A subset S of $V(G)$ is a dominating set of G if for every $v \in V(G) \setminus S$, there exists $x \in S$ such that $xv \in E(G)$. A set $S \subseteq V(G)$ is said to be an outer-connected dominating set in G if S is dominating and either $S = V(G)$ or $\langle V(G) \setminus S\rangle$ is connected. A fair dominating set in graph G is a dominating set S such that all vertices in $V(G) \setminus S$ are dominated by the equal number of vertices in S. A nonempty subset $S \subseteq V(G)$ is an outer-connected fair dominating set of G, if S is a fair dominating set of G and the subgraph $\langle V(G) \setminus S\rangle$ induced by $V(G) \setminus S$ is connected. The outer-connected fair domination number of G is the minimum cardinality of an outer-connected fair dominating set of G, denoted by $\tilde{\gamma}_{cfd}(G)$.In this paper, we give the outer-connected fair dominating set and the outer-connected fair domination number in the join of two connected noncomplete graphs.

Keywords: corona, dominating set, join, fair, outer-connected

I. Introduction

A graph G is a pair $(V(G), E(G))$, where $V(G)$ is a finite nonempty set called the vertex-set of G and $E(G)$ is a set of unordered pairs $\{u, v\}$ (or simply uv) of distinct elements from $V(G)$ called the edge-set of G. The elements of $V(G)$ are called vertices and the cardinality $|V(G)|$of $V(G)$ is the order of G. The elements of $E(G)$ are called edges and the cardinality $|E(G)|$ of $E(G)$ is the size of G. If $|V(G)| = 1$, then G is called a trivial graph. If$E(G) = \emptyset$, then G is called an empty graph. The open neighborhood of a vertex $v \in V(G)$ is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The elements of $N_G(v)$ are called neighbors of v. The closed neighborhood of $v \in V(G)$is the set $N_G[v] = N_G(v) \cup \{v\}$. If $X \subseteq V(G)$, the open neighborhood of X in G is the set $N_G(X) = \bigcup_{v\in X} N_G(v)$. The closed neighborhood of X in G is the set $N_G[X] = \bigcup_{v\in X} N_G[v] = N_G(X) \cup X$. When no confusion arises, $N_G[x]$ [res. $N_G(x)$] will be denoted by $N[x]$ [resp. $N(x)$]. Au-v walk in G is a sequence of vertices in G, beginning with u and ending at v such that consecutive vertices in the sequence are adjacent. Au-v walk in a graph in which no vertices are repeated is a u-v path. If G contains a u-$v$path, then u and v are said to be connected and u is connected to v (and v is connected to u). A graph G is connected if every two vertices of G are connected, that is, if G contains a u-v path for every pair u, v of vertices of G. Since every vertex is connected to itself, the trivial graph is connected. For the general terminology in graph theory, readers may refer to [1].

Domination in graph was introduced by Claude Berge in 1958 and Oystein Ore in 1962 [2]. Following an article [3] by Ernie Cockayne and Stephen Hedetniemi in 1977, the domination in graphs became an area of study by many researchers. A subset S of $V(G)$ is a dominating set of Gif for every $v \in V(G) \setminus S$, there exists $x \in S$ such that $xv \in E(G)$, that is, $N[S] = V(G)$. The domination number $\gamma(G)$ of G is the smallest cardinality of a dominating set of G. Some studies on domination in graphs were found in the papers [4-17].

A set S of vertices of a graph G is an outer-connected dominating set if every vertex not in S is adjacent to some vertex in S and the sub-graph induced by$V(G) \setminus S$ is connected. The outer-connected domination number $\tilde{\gamma}_c(G)$ is the minimum cardinality of the outer-connected dominating set S of a graph G. The concept of outer-connected domination in graphs was introduced by Cyman [18]. Some related studies of outer-connected domination in graphs are found in [19-28].

A fair dominating set in graph G is a dominating set S such that all vertices in $V(G) \setminus S$ are dominated by the equal number of vertices in S.The fair domination number of G is the minimum cardinality of a fair dominating set of G, denoted by $\gamma_{fd}(G)$. The concepts of fair domination in graphs were introduced by Caro, Hansberg, and Henning [29]. Some related studies of fair domination in graphs are found in [30-35].

Motivated by the introduction of the outer-connected dominating sets and the fair dominating sets, a new variant of domination in graphs is introduced in this paper. Let $G = (V(G), E(G))$ be a nontrivial connected simple graph. A nonempty subset $S \subseteq V(G)$ is an outer-connected fair dominating set of G, if S is a fair dominating set of G and the subgraph $\langle V(G) \setminus S\rangle$ induced by $V(G) \setminus S$ is connected. The outer-connected fair domination number of G is the minimum cardinality of an outer-connected fair dominating set of G, denoted by

$\tilde{\gamma}_{cfd}$(G).In this paper, we give the outer-connected fair dominating set and the outer-connected fair domination number in the join of two connected noncomplete graphs.

II. RESULTS

Definition 2.1 The join of two graphs G and H is the graph $G + H$ with vertex-set $V(G + H) = V(G) \cup V(H)$ and edge-set $E(G + H) = E(G) \cup E(H) \cup \{uv: u \in V(G),\ v \in V(H)\}$.
The following results are needed for the proofs of our subsequent theorem, an outer-connected fair dominating sets in the join of two noncomplete graphs G and H.

Lemma 2.2 Let G and H be connected non-complete graphs. If$S_G \subset V(G)$ is a fair dominating set of G, $\langle V(G) \setminus S_G \rangle$ is connected, and $S_H = V(H)$, then the set $S = S_G \cup S_H \subset V(G + H)$ is an outer-connected fair dominating set of $G + H$.

Proof. Let G and H be connected noncomplete graphs. If$S_G \subset V(G)$ is a fair dominating set of G, then $|N_G(u) \cap S_G| = |N_G(u') \cap S_G|$for all $u, u' \in V(G) \setminus S_G$. Thus, if $S_H = V(H)$,

$$\begin{aligned}
|N_{G+H}(u) \cap S| &= |(N_G(u) \cup V(H)) \cap S| \\
&= |(N_G(u) \cap S) \cup (V(H) \cap S)| \\
&= |[N_G(u) \cap (S_G \cup S_H)] \cup [V(H) \cap (S_G \cup S_H)]| \\
&= |[(N_G(u) \cap S_G) \cup (N_G(u) \cap S_H)] \cup [(V(H) \cap S_G) \cup (V(H) \cap S_H)]| \\
&= |[(N_G(u) \cap S_G) \cup \emptyset] \cup [\emptyset \cup S_H]| \\
&= |(N_G(u) \cap S_G) \cup S_H| \\
&= |(N_G(u) \cap S_G) \cup S_H| \\
&= |(N_G(u') \cap S_G) \cup S_H|, \text{ note that } |N_G(u) \cap S_G| = |N_G(u') \cap S_G| \\
&= |(N_G(u') \cup S_H) \cap (S_G \cup S_H)| \\
&= |(N_G(u') \cup V(H)) \cap S| \\
&= |N_{G+H}(u') \cap S|
\end{aligned}$$

That is, $|N_{G+H}(u) \cap S| = |N_{G+H}(u') \cap S|$ for all $u, u' \in V(G) \setminus S_G$ or $u, u' \in V(G + H) \setminus S$. This implies that $S \subset V(G + H)$ is a fair dominating set of $G + H$. Since $\langle V(G) \setminus S_G \rangle$ is connected, it follows that

$$\begin{aligned}
\langle V(G + H) \setminus S \rangle &= \langle \big(V(G) \cup V(H)\big) \setminus S \rangle \\
&= \langle (V(G) \setminus S) \cup (V(H) \setminus S) \rangle \\
&= \langle (V(G) \setminus S_G) \cup [V(H) \setminus (S_G \cup S_H)] \rangle \\
&= \langle \big(V(G) \setminus S_G) \cup [V(H) \setminus (S_G \cup V(H)\big)] \rangle \\
&= \langle (V(G) \setminus S_G) \cup \emptyset \rangle \\
&= \langle V(G) \setminus S_G \rangle \text{ is connected.}
\end{aligned}$$

That is, the subgraph induced by $V(G + H) \setminus S$ is connected. Thus, $S = S_G \cup S_H \subset V(G + H)$ is an outer-connected fair dominating set of $G + H$. ■

Lemma 2.3 Let G and H be connected non-complete graphs. If$S_G = V(G)$, $\langle V(H) \setminus S_H \rangle$ is connected, and $S_H \subset V(H)$is a fair dominating set of H, then the set $S = S_G \cup S_H \subset V(G + H)$ is an outer-connected fair dominating set of $G + H$.

Proof. Let G and H be connected non-complete graphs. If$S_H \subset V(H)$ is a fair dominating set of H, then $|N_H(v) \cap S_H| = |N_H(v') \cap S_H|$ for all $v, v' \in V(H) \setminus S_H$. Thus, if $S_G = V(G)$,

$$\begin{aligned}
|N_{G+H}(v) \cap S| &= |(N_H(v) \cup V(G)) \cap S| \\
&= |(N_H(v) \cap S) \cup (V(G) \cap S)| \\
&= |[N_H(v) \cap (S_G \cup S_H)] \cup [V(G) \cap (S_G \cup S_H)]| \\
&= |[(N_H(v) \cap S_G) \cup (N_H(v) \cap S_H)] \cup [(V(G) \cap S_G) \cup (V(G) \cap S_H)]| \\
&= |\emptyset \cup (N_H(v) \cap S_H)] \cup [S_G \cup \emptyset]| \\
&= |(N_H(v) \cap S_H)] \cup S_G| \\
&= |(N_H(v') \cap S_H)] \cup S_G|, \text{ note that } |N_H(v) \cap S_H| = |N_H(v') \cap S_H| \\
&= |(N_H(v') \cup S_G) \cap (S_H \cup S_G)|
\end{aligned}$$

$$\begin{aligned} &= |(N_H(v') \cup V(G)) \cap S| \\ &= |N_{G+H}(v') \cap S| \end{aligned}$$

That is, $|N_{G+H}(v) \cap S| = |N_{G+H}(v') \cap S|$ for all $v, v' \in V(H) \setminus S_H$ or $v, v' \in V(G+H) \setminus S$. This implies that $S \subset V(G+H)$ is a fair dominating set of $G+H$. Since $\langle V(H) \setminus S_H \rangle$ is connected, it follows that

$$\begin{aligned} \langle V(G+H) \setminus S \rangle &= \langle (V(G) \cup V(H)) \setminus S \rangle \\ &= \langle (V(G) \setminus S) \cup (V(H) \setminus S) \rangle \\ &= \langle [(V(G) \setminus (S_G \cup S_H))] \cup (V(H) \setminus S_H) \rangle \\ &= \langle [(V(G) \setminus (V(G) \cup S_H))] \cup (V(H) \setminus S_H) \rangle \\ &= \langle \emptyset \cup (V(H) \setminus S_H) \rangle \\ &= \langle V(H) \setminus S_H \rangle \text{ is connected.} \end{aligned}$$

That is, the subgraph induced by $V(G+H) \setminus S$ is connected. Thus, $S = S_G \cup S_H \subset V(G+H)$ is an outer-connected fair dominating set of $G+H$. ■

Lemma 2.4 Let G and H be connected non-complete graphs. If $S_G \subset V(G)$ is a fair dominating set of G, $S_H \subset V(H)$ is a fair dominating set of H, and $|N_G(u) \cap S_G| + |S_H| = |N_H(v) \cap S_H| + |S_G|$, then the set $S = S_G \cup S_H \subset V(G+H)$ is an outer-connected fair dominating set of $G+H$.

Proof. Let G and H be connected non-complete graphs. Suppose that $x, y \in V(G+H) \setminus S$.

Case 1. Consider $x, y \in V(G) \setminus S_G$. Let $x = u$ and $y = u'$. Since $S_G \subset V(G)$ is a fair dominating set of G, for each $u, u' \in V(G) \setminus S_G$, $|N_G(u) \cap S_G| = |N_G(u') \cap S_G|$. Thus,

$$\begin{aligned} |N_{G+H}(u) \cap S| &= |[N_G(u) \cup V(H)] \cap S| \\ &= |[N_G(u) \cap S] \cup [V(H) \cap S]| \\ &= |[N_G(u) \cap (S_G \cup S_H)] \cup [V(H) \cap (S_G \cup S_H)]| \\ &= |[(N_G(u) \cap S_G) \cup (N_G(u) \cap S_H)] \cup [(V(H) \cap S_G) \cup (V(H) \cap S_H)]| \\ &= |[(N_G(u) \cap S_G) \cup S_H] \cup [\emptyset \cup S_H]| \\ &= |(N_G(u) \cap S_G) \cup S_H| \\ &= |(N_G(u') \cap S_G) \cup S_H| \\ &= |[(N_G(u') \cap S_G) \cup S_H] \cup [\emptyset \cup S_H]| \\ &= |[(N_G(u') \cap S_G) \cup (N_G(u') \cap S_H)] \cup [(V(H) \cap S_G) \cup (V(H) \cap S_H)]| \\ &= |[N_G(u') \cap (S_G \cup S_H)] \cup [V(H) \cap (S_G \cup S_H)]| \\ &= |[N_G(u') \cap S] \cup [V(H) \cap S]| \\ &= |[N_G(u') \cup V(H)] \cap S| \\ &= |N_{G+H}(u') \cap S| \end{aligned}$$

That is, $|N_{G+H}(u) \cap S| = |N_{G+H}(u') \cap S|$ for all $u, u' \in V(G) \setminus S_G$ or $u, u' \in V(G+H) \setminus S$. This implies that $S \subset V(G+H)$ is a fair dominating set of $G+H$, if $x = u, y = u' \in V(G) \setminus S_G$. Since $uv \in E(G+H)$ for all $u \in V(G) \setminus S_G$ and for all $v \in V(H) \setminus S_H$, it follows that $\langle V(G+H) \setminus S \rangle$ is connected. Hence, $S = S_G \cup S_H \subset V(G+H)$ is an outer-connected fair dominating set of $G+H$, if $x = u, y = u' \in V(G) \setminus S_G$.

Case 2. Consider $x, y \in V(H) \setminus S_H$. Let $x = v$ and $y = v'$. Since $S_H \subset V(H)$ is a fair dominating set of H, then for each $v, v' \in V(H) \setminus S_H$, $|N_H(v) \cap S_H| = |N_H(v') \cap S_H|$. Thus,

$$\begin{aligned} |N_{G+H}(v) \cap S| &= |[N_H(v) \cup V(G)] \cap S| \\ &= |[N_H(v) \cap S] \cup [V(G) \cap S]| \\ &= |[N_H(v) \cap (S_G \cup S_H)] \cup [V(G) \cap (S_G \cup S_H)]| \\ &= |[(N_H(v) \cap S_G) \cup (N_H(v) \cap S_H)] \cup [(V(G) \cap S_G) \cup (V(G) \cap S_H)]| \\ &= |[S_G \cup (N_H(v) \cap S_H)] \cup [S_G \cup \emptyset]| \\ &= |(N_H(v) \cap S_H) \cup S_G| \\ &= |(N_H(v') \cap S_H) \cup S_G| \\ &= |[(N_H(v') \cap S_H) \cup S_G] \cup [S_G \cup \emptyset]| \\ &= |[(N_H(v') \cap S_H) \cup (N_H(v') \cap S_G)] \cup [(V(G) \cap S_G) \cup (V(G) \cap S_H)]| \end{aligned}$$

$$
\begin{aligned}
&= |[N_H(v') \cap (S_H \cup S_G)] \cup [V(G) \cap (S_G \cup S_H)]| \\
&= |[N_H(v') \cap S] \cup [V(G) \cap S]| \\
&= |[N_H(v') \cup V(G)] \cap S| \\
&= |N_{G+H}(v') \cap S|
\end{aligned}
$$

That is, $|N_{G+H}(v) \cap S| = |N_{G+H}(v') \cap S|$ for all $v, v' \in V(H) \setminus S_H$or $v, v' \in V(G+H) \setminus S$. This implies that $S \subset V(G+H)$ is a fair dominating set of $G+H$, if $x = v, y = v' \in V(H) \setminus S_H$ Since $uv \in E(G+H)$ for all $u \in V(G) \setminus S_G$ and $v \in V(H) \setminus S_H$, it follows that $\langle V(G+H) \setminus S \rangle$ is connected. Hence, $S = S_G \cup S_H \subset V(G+H)$ is an outer-connected fair dominating set of $G+H$, if $x = v, y = v' \in V(H) \setminus S_H$.

Case 3. Consider $x \in V(G) \setminus S_G$ and $y \in V(H) \setminus S_H$. Let $x = u$ and $y = v$. Then for each$u \in V(G) \setminus S_G$ and $v \in V(H) \setminus S_H$,$|N_G(u) \cap S_G| + |S_H| = |N_H(v) \cap S_H| + |S_G|$. Thus,

$$
\begin{aligned}
|N_{G+H}(u) \cap S| &= |[N_G(u) \cup V(H)] \cap S| \\
&= |[N_G(u) \cap S] \cup [V(H) \cap S]| \\
&= |[N_G(u) \cap (S_G \cup S_H)] \cup [V(H) \cap (S_G \cup S_H)]| \\
&= |[(N_G(u) \cap S_G) \cup (N_G(u) \cap S_H)] \cup [(V(H) \cap S_G) \cup (V(H) \cap S_H)]| \\
&= |[(N_G(u) \cap S_G) \cup S_H] \cup [\emptyset \cup S_H]| \\
&= |(N_G(u) \cap S_G) \cup S_H| \\
&= |(N_G(u) \cap S_G)| + |S_H| \\
&= |(N_H(v) \cap S_H)| + |S_G| \\
&= |(N_H(v) \cap S_H) \cup S_G| \\
&= |[(N_H(v) \cap S_H) \cup S_G] \cup [\emptyset \cup S_G]| \\
&= |[(N_H(v) \cap S_H) \cup (N_H(v) \cap S_G)] \cup [(V(G) \cap S_H) \cup (V(G) \cap S_G)]| \\
&= |[N_H(v) \cap (S_H \cup S_G)] \cup [V(G) \cap (S_H \cup S_G)]| \\
&= |[N_H(v) \cap S] \cup [V(G) \cap S]| \\
&= |[N_H(v) \cup V(G)] \cap S| \\
&= |N_{G+H}(v) \cap S|
\end{aligned}
$$

That is, $|N_{G+H}(u) \cap S| = |N_{G+H}(v) \cap S|$ for all$u \in V(G) \setminus S_G$and for all $v \in V(H) \setminus S_H$ or $u, v \in V(G+H) \setminus S$. This implies that $S \subset V(G+H)$ is a fair dominating set of $G+H$, if $x = u \in V(G) \setminus S_G$ and $y = v \in V(H) \setminus S_H$. Since $uv \in E(G+H)$ for all $u \in V(G) \setminus S_G$ and $v \in V(H) \setminus S_H$, it follows that $\langle V(G+H) \setminus S \rangle$ is connected. Hence, $S = S_G \cup S_H \subset V(G+H)$ is an outer-connected fair dominating set of $G+H$, if $x = u \in V(G) \setminus S_G$ and $y = v \in V(H) \setminus S_H$.

Therefore, $S = S_G \cup S_H \subset V(G+H)$ is an outer-connected fair dominating set of $G+H$ for all $x, y \in V(G+H) \setminus S$. ■

Lemma 2.5 Let$G = \overline{K}_m$, $H = \overline{K}_n$, with $m, n \geq 2$. If $S_G \subset V(G)$, $S_H \subset V(H)$, and $|S_G| = |S_H|$, then the set $S = S_G \cup S_H \subset V(G+H)$ is an outer-connected fair dominating set of $G+H$.

Proof. Let$G = \overline{K}_m$, $H = \overline{K}_n$, with $m, n \geq 2$. Suppose that $x, y \in V(G+H) \setminus S$. If $S_G \subset V(G)$ and $S_H \subset V(H)$, then $V(G) \setminus S_G \neq \emptyset$ and $V(H) \setminus S_H \neq \emptyset$.

Case 1. Consider $x, y \in V(G) \setminus S_G$. For $m \geq 2$, $N_G(x) \cap S_H = S_H = N_G(y) \cap S_H$. Thus,

$$
\begin{aligned}
|N_{G+H}(x) \cap S| &= |[N_G(x) \cup V(H)] \cap S| \\
&= |[N_G(x) \cap S] \cup [V(H) \cap S]| \\
&= |[N_G(x) \cap (S_G \cup S_H)] \cup [S_H]| \\
&= |[(N_G(x) \cap S_G) \cup (N_G(x) \cap S_H)] \cup [S_H]| \\
&= |[\emptyset \cup (N_G(y) \cap S_H)] \cup [S_H]| \\
&= |[(N_G(y) \cap S_G) \cup (N_G(y) \cap S_H)] \cup [S_H]| \\
&= |[N_G(y) \cap (S_G \cup S_H)] \cup [S_H]| \\
&= |[N_G(y) \cap S] \cup [V(H) \cap S]| \\
&= |[N_G(y) \cup V(H)] \cap S| \\
&= |N_{G+H}(y) \cap S|
\end{aligned}
$$

That is, $|N_{G+H}(x) \cap S| = |N_{G+H}(y) \cap S|$ for all$x, y \in V(G) \setminus S_G$ or $x, y \in V(G+H) \setminus S$. This implies that $S \subset V(G+H)$ is a fair dominating set of $G+H$, if $x, y \in V(G) \setminus S_G$. Since $uv \in E(G+H)$ for all $u \in V(G) \setminus S_G$ and $v \in V(H) \setminus S_H$, it follows that $\langle V(G+H) \setminus S \rangle$ is connected. Hence, $S = S_G \cup S_H \subset V(G+H)$ is an outer-connected fair dominating set of $G+H$, if $x, y \in V(G) \setminus S_G$.

Case 2. Consider $x, y \in V(H) \setminus S_H$. For $n \geq 2$, $N_H(x) \cap S_G = N_H(y) \cap S_G$. Thus,

$$
\begin{aligned}
|N_{G+H}(x) \cap S| &= |[N_H(x) \cup V(G)] \cap S| \\
&= |[N_H(x) \cap S] \cup [V(G) \cap S]| \\
&= |[N_H(x) \cap (S_G \cup S_H)] \cup [S_G]| \\
&= |[(N_H(x) \cap S_G) \cup (N_H(x) \cap S_H)] \cup [S_G]| \\
&= |[(N_H(y) \cap S_G) \cup \emptyset] \cup [S_G]| \\
&= |[(N_H(y) \cap S_G) \cup (N_H(y) \cap S_H)] \cup [S_G]| \\
&= |[N_H(y) \cap (S_G \cup S_H)] \cup [S_G]| \\
&= |[N_H(y) \cap S] \cup [V(G) \cap S]| \\
&= |[N_H(y) \cup V(G)] \cap S| \\
&= |N_{G+H}(y) \cap S|
\end{aligned}
$$

That is, $|N_{G+H}(x) \cap S| = |N_{G+H}(y) \cap S|$ for all$x, y \in V(H) \setminus S_H$ or $x, y \in V(G+H) \setminus S$. This implies that $S \subset V(G+H)$ is a fair dominating set of $G+H$, if $x, y \in V(H) \setminus S_H$. Since $uv \in E(G+H)$ for all $u \in V(G) \setminus S_G$ and $v \in V(H) \setminus S_H$, it follows that $\langle V(G+H) \setminus S \rangle$ is connected. Hence, $S = S_G \cup S_H \subset V(G+H)$ is an outer-connected fair dominating set of $G+H$, if $x, y \in V(H) \setminus S_H$.

Case 3. Consider $x \in V(G) \setminus S_G$ and $y \in V(H) \setminus S_H$. For $m \geq 2$, $|N_G(x) \cap S_H| = |S_H| = |S_G| = |N_H(y) \cap S_G|$. Thus,

$$
\begin{aligned}
|N_{G+H}(x) \cap S| &= |[N_G(x) \cup V(H)] \cap S| \\
&= |[N_G(x) \cap S] \cup [V(H) \cap S]| \\
&= |[N_G(x) \cap (S_G \cup S_H)] \cup [V(H) \cap S]| \\
&= |[(N_G(x) \cap S_G) \cup (N_G(x) \cap S_H)] \cup [V(H) \cap S]| \\
&= |N_G(x) \cap S_G| + |N_G(x) \cap S_H| + |V(H) \cap S| \\
&= |\emptyset| + |S_H| + |V(H) \cap S| \\
&= |\emptyset| + |S_G| + |V(H) \cap S| \\
&= |N_H(y) \cap S_H| + |N_H(y) \cap S_G| + |V(H) \cap S| \\
&= |[(N_H(y) \cap S_H) \cup (N_H(y) \cap S_G)] \cup [V(H) \cap S]| \\
&= |[N_H(y) \cap (S_H \cup S_G)] \cup [V(H) \cap S]| \\
&= |[N_H(y) \cap S] \cup [V(H) \cap S]| \\
&= |[N_G(y) \cup V(H)] \cap S| \\
&= |N_{G+H}(y) \cap S|
\end{aligned}
$$

That is, $|N_{G+H}(x) \cap S| = |N_{G+H}(y) \cap S|$ for all$x \in V(G) \setminus S_G$ and $y \in V(H) \setminus S_H$(or $x, y \in V(G+H) \setminus S$). This implies that $S \subset V(G+H)$ is a fair dominating set of $G+H$, if $x \in V(G) \setminus S_G$ and $y \in V(H) \setminus S_H$. Since $uv \in E(G+H)$ for all $u \in V(G) \setminus S_G$ and $v \in V(H) \setminus S_H$, it follows that $\langle V(G+H) \setminus S \rangle$ is connected. Hence, $S = S_G \cup S_H \subset V(G+H)$ is an outer-connected fair dominating set of $G+H$, if$x \in V(G) \setminus S_G$ and $y \in V(H) \setminus S_H$.

Therefore, $S = S_G \cup S_H \subset V(G+H)$ is an outer-connected fair dominating set of $G+H$ for all $x, y \in V(G+H) \setminus S$. ■

Lemma 2.6 Let G and H be connected noncomplete graphs. If for every $u \in S \subset V(G)$, $uu^{'} \in E(G)$ for all $u^{'} \in V(G) \setminus S$, then S is an outer-connected fair dominating set of $G+H$.

Proof. Let G and H be connected noncomplete graphs of orders m and n respectively. Then $m \geq 3$ and $n \geq 3$. Suppose that for every $u \in S \subset V(G)$, $uu^{'} \in E(G)$ for all $u^{'} \in V(G) \setminus S$. Then $N_G(u') \cap S = S$ for all $u^{'} \in V(G) \setminus S$. Let $x, y \in V(G+H) \setminus S$.

Case 1. Consider $x, y \in V(G) \setminus S$. For $m \geq 3$, $N_G(x) \cap S = S = N_G(y) \cap S$. Thus,

$$\begin{aligned}|N_{G+H}(x) \cap S| &= |[N_G(x) \cup V(H)] \cap S| \\ &= |[N_G(x) \cap S] \cup [V(H) \cap S]| \\ &= |[N_G(y) \cap S] \cup [V(H) \cap S]| \\ &= |[N_G(y) \cup V(H)] \cap S| \\ &= |N_{G+H}(y) \cap S|\end{aligned}$$

That is, $|N_{G+H}(x) \cap S| = |N_{G+H}(y) \cap S|$ for all$x, y \in V(G) \setminus S$. This implies that $S \subset V(G+H)$ is a fair dominating set of $G+H$, if $x, y \in V(G) \setminus S$.

Case 2. Consider $x, y \in V(H)$. For $n \geq 3$, $N_{G+H}(x) \cap S = S = N_{G+H}(y) \cap S$. Thus, $|N_{G+H}(x) \cap S| = |N_{G+H}(y) \cap S|$ for all $x, y \in V(H)$. This implies that $S \subset V(G+H)$ is a fair dominating set of $G+H$, if $x, y \in V(H)$.

Case 3. Consider $x \in V(G) \setminus S$ and $y \in V(H)$. For $m, n \geq 3$, $N_G(x) \cap S = S = [N_H(y) \cup V(G)] \cap S$. Thus,

$$\begin{aligned}|N_{G+H}(x) \cap S| &= |[N_G(x) \cup V(H)] \cap S| \\ &= |[N_G(x) \cap S] \cup [V(H) \cap S]| \\ &= |[(N_H(y) \cup V(G)) \cap S] \cup [\emptyset]| \\ &= |N_{G+H}(y) \cap S|\end{aligned}$$

That is, $|N_{G+H}(x) \cap S| = |N_{G+H}(y) \cap S|$ for all$x \in V(G) \setminus S$ and $y \in V(H)$. This implies that $S \subset V(G+H)$ is a fair dominating set of $G+H$, if $x \in V(G) \setminus S$ and $y \in V(H)$.

Therefore, using Case 1, Case 2, and Case 3, $|N_{G+H}(x) \cap S| = |N_{G+H}(y) \cap S|$ for all$x, y \in V(G+H) \setminus S$. That is $S \subset V(G+H)$ is a fair dominating set of $G+H$. Since, $uv \in E(G+H)$ for all $u \in V(G) \setminus S$ and $v \in V(H)$, it follows that $\langle V(G+H) \setminus S \rangle$ is connected. Thus, S is an outer-connected fair dominating set of $G+H$.■

Lemma 2.7 Let G and H be connected noncomplete graphs. If for every $v \in S \subset V(H)$, $vv^{'} \in E(H)$ for all $v^{'} \in V(H) \setminus S$, then S is an outer-connected fair dominating set of $G+H$.

Proof. Let G and H be connected noncomplete graphs of orders m and n respectively. Then $m \geq 3$ and $n \geq 3$. Suppose that for every $v \in S \subset V(H)$, $vv^{'} \in E(H)$ for all $v^{'} \in V(H) \setminus S$. Then $N_H(v') \cap S = S$ for all $v^{'} \in V(H) \setminus S$. Let $x, y \in V(G+H) \setminus S$.

Case 1. Consider $x, y \in V(G)$. For $m \geq 3$, $N_{G+H}(x) \cap S = S = N_{G+H}(y) \cap S$. Thus, $|N_{G+H}(x) \cap S| = |N_{G+H}(y) \cap S|$ for all $x, y \in V(G)$. This implies that $S \subset V(G+H)$ is a fair dominating set of $G+H$, if $x, y \in V(G)$.

Case 2. Consider $x, y \in V(H) \setminus S$. For $m \geq 3$, $N_H(x) \cap S = S = N_H(y) \cap S$. Thus,

$$\begin{aligned}|N_{G+H}(x) \cap S| &= |[N_H(x) \cup V(G)] \cap S| \\ &= |[N_H(x) \cap S] \cup [V(G) \cap S]| \\ &= |[N_H(y) \cap S] \cup [V(G) \cap S]| \\ &= |[N_H(y) \cup V(G)] \cap S| \\ &= |N_{G+H}(y) \cap S|\end{aligned}$$

That is, $|N_{G+H}(x) \cap S| = |N_{G+H}(y) \cap S|$ for all$x, y \in V(H) \setminus S$. This implies that $S \subset V(G+H)$ is a fair dominating set of $G+H$, if $x, y \in V(H) \setminus S$.

Case 3. Consider $x \in V(G)$ and $y \in V(H) \setminus S$. For $m, n \geq 3$, $[N_G(x) \cup V(H)] \cap S = S = N_H(y) \cap S$. Thus,

$$\begin{aligned}|N_{G+H}(x) \cap S| &= |[N_G(x) \cup V(H)] \cap S| \\ &= |N_H(y) \cap S| \\ &= |[N_H(y) \cap S] \cup [\emptyset]| \\ &= |[N_H(y) \cap S] \cup [V(G) \cap S]| \\ &= |[N_H(y) \cup V(G)] \cap S| \\ &= |N_{G+H}(y) \cap S|\end{aligned}$$

That is, $|N_{G+H}(x) \cap S| = |N_{G+H}(y) \cap S|$ for all$x \in V(G)$ and $y \in V(H) \setminus S$. This implies that $S \subset V(G+H)$ is a fair dominating set of $G+H$, if $x \in V(G)$ and $y \in V(H) \setminus S$.

Therefore, using Case 1, Case 2, and Case 3, $|N_{G+H}(x) \cap S| = |N_{G+H}(y) \cap S|$ for all$x, y \in V(G+H) \setminus S$. That is $S \subset V(G+H)$ is a fair dominating set of $G+H$. Since, $uv \in E(G+H)$ for all $u \in V(G) \setminus S$ and $v \in V(H)$, it follows that $\langle V(G+H) \setminus S \rangle$ is connected. Thus, S is an outer-connected fair dominating set of $G+H$.■

The next result, gives an outer-connected fair dominating sets in the join of two noncomplete graphs G and H.

Theorem 2.8 Let G and H be connected noncomplete graphs of order m and n respectively. The set $S = S_G \cup S_H \subset V(G+H)$ is an outer-connected fair dominating set of $G+H$if one of the following is satisfied.

i) $S_G \subset V(G)$ is a fair dominating set of G, $\langle V(G) \setminus S_G \rangle$ is connected, and $S_H = V(H)$.
ii) $S_G = V(G)$, $\langle V(H) \setminus S_H \rangle$ is connected, and $S_H \subset V(H)$ is a fair dominating set of H.
iii) $S_G \subset V(G)$ is a fair dominating set of G, $S_H \subset V(H)$ is a fair dominating set of H, and $|N_G(u) \cap S_G| + |S_H| = |N_H(v) \cap S_H| + |S_G|$,
iv) $G = \overline{K}_m$, $H = \overline{K}_n$, $m, n \geq 2$,$S_G \subset V(G)$, $S_H \subset V(H)$, and $|S_G| = |S_H|$.
v) For every $u \in S \subset V(G)$, $uu' \in E(G)$ for all $u' \in V(G) \setminus S$.
vi) For every $v \in S \subset V(H)$, $vv' \in E(H)$ for all $v' \in V(H) \setminus S$.

Proof. i) Suppose that $S_G \subset V(G)$ is a fair dominating set of G, $\langle V(G) \setminus S_G \rangle$ is connected, and $S_H = V(H)$. Then the set $S = S_G \cup S_H \subset V(G+H)$ is an outer-connected fair dominating set of $G+H$ by Lemma 2.2.

ii) Suppose that $S_G = V(G)$, $\langle V(H) \setminus S_H \rangle$ is connected, and $S_H \subset V(H)$ is a fair dominating set of H. Then the set $S = S_G \cup S_H \subset V(G+H)$ is an outer-connected fair dominating set of $G+H$ by Lemma 2.3.

iii) Suppose that $S_G \subset V(G)$ is a fair dominating set of G, $S_H \subset V(H)$ is a fair dominating set of H, and $|N_G(u) \cap S_G| + |S_H| = |N_H(v) \cap S_H| + |S_G|$. Then the set $S = S_G \cup S_H \subset V(G+H)$ is an outer-connected fair dominating set of $G+H$ by Lemma 2.4.

iv) Suppose that $G = \overline{K}_m$, $H = \overline{K}_n$, $m, n \geq 2$, $S_G \subset V(G)$, $S_H \subset V(H)$, and $|S_G| = |S_H|$. Then the set $S = S_G \cup S_H \subset V(G+H)$ is an outer-connected fair dominating set of $G+H$by Lemma 2.5.

v) Suppose that for every $u \in S \subset V(G)$, $uu' \in E(G)$ for all $u' \in V(G) \setminus S$. Then the set $S \subset V(G+H)$ is an outer-connected fair dominating set of $G+H$ by Lemma 2.6.

vi) Suppose that for every $v \in S \subset V(H)$, $vv' \in E(H)$ for all $v' \in V(H) \setminus S$. Then the set $S \subset V(G+H)$ is an outer-connected fair dominating set of $G+H$ by Lemma 2.7.

This completes the proofs. ■

The next result is an immediate consequence of Theorem 2.8.

Corollary 2.9 Let G and H be noncomplete graphs of order $m \geq 3$ and $n \geq 3$ respectively. Then

$$\tilde{\gamma}_{cfd}(G+H) = \begin{cases} 1 & if\ \gamma(G) = 1 \text{ or } \gamma(H) = 1 \\ 2 & if\ \mathrm{G} = \overline{K}_m\ and\ H = \overline{K}_n \end{cases}.$$

Proof. Let G and H be noncomplete graphs of order $m \geq 3$ and $n \geq 3$ respectively.

Suppose that $\gamma(G) = 1$. Let $S = \{u\} \subset \mathrm{V(G)}$ be a dominating set of G. Since $u \in S$, $uu' \in E(G)$ for all $u' \in V(G) \setminus S$, it follows that the set $S \subset V(G+H)$ is an outer-connected fair dominating set of $G+H$ by Theorem 2.8(v). Thus, $\tilde{\gamma}_{cfd}(G+H) \leq |S| = |\{u\}| = 1$. Since $\tilde{\gamma}_{cfd}(G+H) \geq 1$, it follows that $\tilde{\gamma}_{cfd}(G+H) = 1$. Similarly, if $\gamma(H) = 1$, then using Theorem 2.8(vi), $\tilde{\gamma}_{cfd}(G+H) = 1$.

Suppose that $G = \overline{K}_m$ and $H = \overline{K}_n$ for $m, n > 2$. Let $S_G = \{u\} \subset V(G)$, $S_H = \{v\} \subset V(H)$. Then $|S_G| = |S_H|$. Using Theorem 2.8(iv), the set $S = S_G \cup S_H \subset V(G + H)$ is an outer-connected fair dominating set of $G + H$. Thus, $\tilde{\gamma}_{cfd}(G + H) \leq |S| = |\{u, v\}| = 2$. Since if for any $x \in V(G + H)$, $S' = \{x\}$ is not a dominating set of $G + H$, it follows that $\tilde{\gamma}_{cfd}(G + H) \neq 1$. Hence, $2 \leq \tilde{\gamma}_{cfd}(G + H) \leq |S| = |\{u, v\}| = 2$, that is, $\tilde{\gamma}_{cfd}(G + H=2$.

This completes the proofs. ■

III. Conclusion

In this work, we introduced a new parameter of domination in graphs - the outer-connected fair domination in graphs. Some results on the outer-connected fair dominating set in the join of two connected noncomplete graphs were proven. The outer-connected fair domination number in the join of two graphs were computed. This study will pave a way to new research such bounds and other binary operations of two graphs. Other parameters involving outer-connected fair domination in graphs may also be explored. Finally, the characterization of an outer-connected fair domination in graphs and its bounds is a promising extension of this study.

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