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Secure Fair Domination in the Join of Two Graphs

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Abstract: Let G be a connected simple graph. A dominating set $S \subset V(G)$ is a fair dominating set in G if $S = V(G)$ or if $S \neq V(G)$ and all vertices not in S are dominated by the same number of vertices from S, that is, $|N(u) \cap S| = |N(v) \cap S| > 0$ for every two vertices $u, v \in V(G) \setminus S$.A fair dominating set S of V(G) is a secure fair dominating set of G if for each $u \in V(G) \setminus S$, there exists $v \in S$ such that $uv \in E(G)$ and the set $(S \setminus \{v\}) \cup \{u\}$ is a fair dominating set of G. The minimum cardinality of a secure fair dominating set of G, denoted by $\gamma_{sfd}(G)$, is called the secure fair domination number of G. In this paper, we give some results on the secure fair domination in the join of two nontrivial connected graphs.

Keywords: dominating set, secure dominating set, fair dominating set, secure fair dominating set, join of two graphs

I. INTRODUCTION

In [1], Claude Berge and Oystein Ore introduced the domination in graph. Claude Berge, a French mathematician, and Oystein Ore, a Norwegian-American mathematician, are considered pioneers of graph theory, particularly in the area of domination theory. Berge introduced the concept of the "coefficient of external stability" (now called the domination number) in 1958, while Ore formalized "dominating sets" and the domination number in 1962, building on Berge's work. "Towards a Theory of Domination in Graphs" [2], is a seminal 1977 paper by Cockayne and Hedetniemi that laid the groundwork for the study of domination in graphs. It introduced key concepts like dominating sets, the domatic number, and the relationship between domination and graph colorings, providing a foundational framework for later research on network analysis, optimization, and other applications The contributions of Claude Berge and Oystein Ore, Cockayne and Hedetniemi in the area of domination in graphs became an area of study by many researchers [3 - 18].

Secure domination in graphs was studied and introduced by E.J. Cockayne et.al [19, 20]. Accordingly, a dominating set S of $V(G)$ is a secure dominating set of G if for each $u \in V(G) \setminus S$, there exists $v \in S$ such that $uv \in E(G)$ and the set$(S \setminus \{v\}) \cup \{u\}$ is a dominating set of G. The minimum cardinality of a secure dominating set of G, denoted by $\gamma_s(G)$, is called the secure domination number of G. In [16], Enriquez and Canoy introduced a variant of secure domination in graphs, the concept of secure convex domination in graphs. Some studies on secure domination in graphs were found in the paper [21 - 28].

In 2011, Caro, Hansberg and Henning [29] introduced the fair domination and k-fair domination in graphs. A dominating subset S of $V(G)$ is a fair dominating set in G if all the vertices not in S are dominated by the same number of vertices from S, that is, $|N(u) \cap S| = |N(v) \cap S|$ for every two distinct vertices u and v from $V(G) \setminus S$ and a subset S of $V(G)$ is a k-fair dominating set in G if for every vertex $v \in V(G) \setminus S$, $|N(v) \cap S| = k$. The minimum cardinality of a fair dominating set of G, denoted by $\gamma_{fd}(G)$, is called the fair domination number of G. A fair dominating set of cardinalities$\gamma_{fd}(G)$ is called γ_{fd}-set. Some studies on fair domination in graphs were found in the paper [30 - 41].

For the general concepts, the reader may refer to [42]. Let $G = (V(G), E(G))$ be a connected simple graph and $v \in V(G)$. The neighborhoodof v is the set $N_G(v) = N(v) = \{u \in V(G): uv \in E(G)\}$. If $S \subseteq V(G)$, thenthe *open neighborhood* of S is the set$N_G(S) = N(S) = \bigcup_{v \in S} N_G(v)$. The *closed neighborhood* of S is$N_G[S] = N[S] = S \cup N(S)$. A subset S of $V(G)$ is a *dominating set* of G if for every $v \in V(G) \setminus S$, there exists $x \in S$ such that $xv \in E(G)$, i.e.,$N[S] = V(G)$. The *domination number*$\gamma(G)$ of G is the smallest cardinality of a dominating set of G.

A fair dominating set S of $V(G)$ is a secure fair dominating set of G if for each $u \in V(G) \setminus S$, there exists $v \in S$ such that $uv \in E(G)$ and the set$(S \setminus \{v\}) \cup \{u\}$ is a fair dominating set of G. The minimum cardinality of a secure fair dominating set of G, denoted by $\gamma_{sfd}(G)$, is called the secure fair domination number of G[43]. In this paper, we give some results on the secure fair domination in the join of two nontrivial connected graphs.

II. RESULTS

Remark 2.1 Let G be a complete graph of order $n \geq 2$. Then a subset $S \subset V(G)$ is a secure fair dominating set of G.

Lemma 2.2 Let G and H be nontrivial connected graphs. Then a nonempty subset $S \subset V(G+H)$ is a secure fair dominating set of $G+H$, if $S = V(G) \cup S_H$ where S_H is a secure fair dominating set of a noncomplete graph H.

Proof: Let G and H be nontrivial connected graphs. Suppose that $S = V(G) \cup S_H$ where S_H is a secure fairdominating set of a noncomplete graph H. Let $u, v \in V(G+H) \setminus S$. Then $u, v \notin S = V(G) \cup S_H$, that is, $u, v \in V(H) \setminus S_H$. Note that for all $u \in V(H) \setminus S_H$, $N_{G+H}(u) = V(G) \cup N_H(u)$. Since$S_H$ is a fair dominating set of H, $|N_H(u) \cap S_H| = |N_H(v) \cap S_H|$. Thus, for all $u, v \in V(G+H) \setminus S$,

$$\begin{aligned}
|N_{G+H}(u) \cap S| &= |[V(G) \cup N_H(u)] \cap [V(G) \cup S_H]| \\
&= |V(G) \cup [N_H(u) \cap S_H]| \\
&= |V(G)| + |N_H(u) \cap S_H| \\
&= |V(G)| + |N_H(v) \cap S_H| \\
&= |V(G) \cup [N_H(v) \cap S_H]| \\
&= |[V(G) \cup N_H(v)] \cap [V(G) \cup S_H]| \\
&= |N_{G+H}(v) \cap S|.
\end{aligned}$$

Hence, $|N_{G+H}(u) \cap S| = |N_{G+H}(v) \cap S|$ for every two distinct vertices u and v from $V(G+H) \setminus S$, that is, S is a fair dominating set of $G+H$.

Now, S_H is a secure fair dominating set of H, implies that S_H is a fair dominating set and for each $y \in V(H) \setminus S_H$, there exists $x \in S_H$ such that $xy \in E(H)$ and the set $(S_H \setminus \{x\}) \cup \{y\}$ is a fair dominating set of H.

Let $y \in V(G+H) \setminus S = \big(V(G) \cup V(H)\big) \setminus (V(G) \cup S_H) = V(H) \setminus S_H$. Then there exists $x \in S_H \subset V(G) \cup S_H = S$ such that $xy \in E(H) \subset E(G+H)$ and the set $(S_H \setminus \{x\}) \cup \{y\} \subset \big((V(G) \cup S_H) \setminus \{x\}\big) \cup \{y\} = (S \setminus \{x\}) \cup \{y\}$ is a fair dominating set of $G+H$.

Hence, S is a fair dominating set and for each $y \in V(G+H) \setminus S$, there exists $x \in S$ such that $xy \in E(G+H)$ and the set $(S \setminus \{x\}) \cup \{y\}$ is a fair dominating set of $G+H$, that is, S is a secure fair dominating set of $G+H$.■

Lemma 2.3 Let G and H be nontrivial connected graphs. Then a nonempty subset $S \subset V(G+H)$ is a secure fair dominating set of $G+H$, if $S = S_G \cup V(H)$ where S_G is a secure fair dominating set of a noncomplete graph G.

Proof: Let G and H be nontrivial connected graphs. Suppose that $S = S_G \cup V(H)$ where S_G is a secure fair dominating set of a noncomplete graph G. Let $u, v \in V(G+H) \setminus S$. Then $u, v \notin S = S_G \cup V(H)$, that is, $u, v \in V(G) \setminus S_G$. Note that for all $u \in V(G) \setminus S_G, N_{G+H}(u) = N_G(u) \cup V(H)$. Since S_G is a fair dominating set of G, $|N_G(u) \cap S_G| = |N_G(v) \cap S_G|$. Thus, for all $u, v \in V(G+H) \setminus S$,

$$\begin{aligned}
|N_{G+H}(u) \cap S| &= |[N_G(u) \cup V(H)] \cap [S_G \cup V(H)]| \\
&= |[N_G(u) \cap S_G] \cup V(H)| \\
&= |N_G(u) \cap S_G| + |V(H)| \\
&= |N_G(v) \cap S_G| + |V(H)| \\
&= |[N_G(v) \cap S_G] \cup V(H)| \\
&= |[N_G(v) \cup V(H)] \cap [S_G \cup V(H)]| \\
&= |N_{G+H}(v) \cap S|.
\end{aligned}$$

Hence, $|N_{G+H}(u) \cap S| = |N_{G+H}(v) \cap S|$ for every two distinct vertices u and v from $V(G+H) \setminus S$, that is, S is a fair dominating set of $G+H$.

Now, S_G is a secure fair dominating set of G, implies that S_G is a fair dominating set and for each $y \in V(G) \setminus S_G$, there exists $x \in S_G$ such that $xy \in E(G)$ and the set $(S_G \setminus \{x\}) \cup \{y\}$ is a fair dominating set of G.

Let $y \in V(G+H) \setminus S = \big(V(G) \cup V(H)\big) \setminus \big(S_G \cup V(H)\big) = V(G) \setminus S_G$. Then there exists $x \in S_G$, that is, $x \in S_G \cup V(H) = S$ such that $xy \in E(G)$, that is, $xy \in E(G+H)$ and the set $(S_G \setminus \{x\}) \cup \{y\}$, that is, $\big((S_G \cup$ $VH \setminus x \cup y = S \setminus x \cup \{y\}$ is a fair dominating set of $G+H$.

Hence, S is a fair dominating set and for each $y \in V(G+H) \setminus S$, there exists $x \in S$ such that $xy \in E(G+H)$ and the set $(S \setminus \{x\}) \cup \{y\}$ is a fair dominating set of $G+H$, that is, S is a secure fair dominating set of $G+H$. ■

Lemma 2.4 Let G and H be nontrivial connected graphs. Then a nonempty subset $S \subset V(G+H)$ is a secure fair dominating set of $G+H$, if S_G and S_H are secure fair dominating sets of noncomplete graphs G and Hrespectively, $|S_G| = |S_H|$, and for each $x \in V(G) \setminus S_G$ and $y \in V(H) \setminus S_H$, there exists $x' \in S_G$ such that $|N_G(x) \cap S_G| = |N_G(x') \cap [(S_G \setminus \{x'\}) \cup \{x\}]|$ and there exists $y' \in S_H$ such that $|N_H(y) \cap S_H| = |N_H(y') \cap [(S_H \setminus \{y'\}) \cup \{y\}]|$.

Proof: Let G and H be nontrivial connected graphs. Suppose that $S = S_G \cup S_H$ where S_G and S_H are secure fair dominating sets of noncomplete graphs G and Hrespectively, $|S_G| = |S_H|$, and for each $x \in V(G) \setminus S_G$ and $y \in V(H) \setminus S_H$, there exists $x' \in S_G$ such that $|N_G(x) \cap S_G| = |N_G(x') \cap [(S_G \setminus \{x'\}) \cup \{x\}]|$ and there exists $y' \in S_H$ such that $|N_H(y) \cap S_H| = |N_H(y') \cap [(S_H \setminus \{y'\}) \cup \{y\}]|$.Let $x, y \in V(G+H) \setminus S$. Consider the following cases.

Case 1. Let $x, y \in V(G) \setminus S_G$. Since S_G is a fair dominating set of G, $|N_G(x) \cap S_G| = |N_G(y) \cap S_G|$ for all $x, y \in V(G) \setminus S_G$. Now,

$$
\begin{aligned}
|N_{G+H}(x) \cap S| &= |[N_G(x) \cup V(H)] \cap (S_G \cup S_H)| \\
&= |[N_G(x) \cap (S_G \cup S_H)] \cup [V(H) \cap (S_G \cup S_H)]| \\
&= |[N_G(x) \cap S_G] \cup S_H| \\
&= |N_G(x) \cap S_G| + |S_H| \\
&= |N_G(y) \cap S_G| + |S_H| \\
&= |[N_G(y) \cap S_G] \cup S_H| \\
&= |[N_G(y) \cap (S_G \cup S_H)] \cup [V(H) \cap (S_G \cup S_H)]| \\
&= |[N_G(y) \cup V(H)] \cap (S_G \cup S_H)| \\
&= N_{G+H}(y) \cap S|.
\end{aligned}
$$

Hence, $|N_{G+H}(x) \cap S| = |N_{G+H}(y) \cap S|$ for every two distinct vertices x and y from $V(G) \setminus S_G$.

Case 2. Let $x, y \in V(H) \setminus S_H$. Since S_H is a fair dominating set of H, $|N_H(x) \cap S_H| = |N_H(y) \cap S_H|$ for all $x, y \in V(H) \setminus S_H$. Now,

$$
\begin{aligned}
|N_{G+H}(x) \cap S| &= |[V(G) \cup N_H(x)] \cap (S_G \cup S_H)| \\
&= |[V(G) \cap (S_G \cup S_H)] \cup [N_H(x) \cap (S_G \cup S_H)]| \\
&= |S_G \cup [N_H(x) \cap S_H]| \\
&= |S_G| + |N_H(x) \cap S_H| \\
&= |S_G| + |N_H(y) \cap S_H| \\
&= |S_G \cup [N_H(y) \cap S_H]| \\
&= |[V(G) \cap (S_G \cup S_H)] \cup [N_H(y) \cap (S_G \cup S_H)]| \\
&= |[V(G) \cup N_H(y)] \cap (S_G \cup S_H)| \\
&= |N_{G+H}(y) \cap S|.
\end{aligned}
$$

Hence, $|N_{G+H}(x) \cap S| = |N_{G+H}(y) \cap S|$ for every two distinct vertices x and y from $V(H) \setminus S_H$.

Case 3. Let $x \in V(G) \setminus S_G$ and $y \in V(H) \setminus S_H$. Given that $|S_G| = |S_H|$ and $|N_G(x) \cap S_G| = |N_H(y) \cap S_H|$,

$$
\begin{aligned}
|N_{G+H}(x) \cap S| &= |[N_G(x) \cup V(H)] \cap (S_G \cup S_H)| \\
&= |[N_G(x) \cap (S_G \cup S_H)] \cup [V(H) \cap (S_G \cup S_H)]| \\
&= |[N_G(x) \cap S_G] \cup S_H|
\end{aligned}
$$

$= |N_G(x) \cap S_G| + |S_H|$
$= |N_H(y) \cap S_H| + |S_G|$
$= |[N_H(y) \cap S_H] \cup S_G|$
$= |[N_H(y) \cap (S_G \cup S_H)] \cup [V(G) \cap (S_G \cup S_H)]|$
$= |[N_H(y) \cup V(G)] \cap (S_G \cup S_H)|$
$= |N_{G+H}(y) \cap S|.$

Hence, $|N_{G+H}(x) \cap S| = |N_{G+H}(y) \cap S|$ for every $x \in V(G) \setminus S_G$ and $y \in V(H) \setminus S_H$.

By Case 1, Case 2, and Case 3, $|N_{G+H}(x) \cap S| = |N_{G+H}(y) \cap S|$ for all $x, y \in V(G+H) \setminus S$. Thus, S is a fair dominating set of $G+H$.

Let $u \in V(G+H) \setminus S$. Clearly, there exists $u^{'} \in S$ such that $uu^{'} \in E(G+H)$ and $(S \setminus \{u^{'}\}) \cup \{u\}$. To show that $S^{'} = (S \setminus \{u^{'}\}) \cup \{u\}$ is a fair dominating set of $G+H$, one of the following is satisfied.

Case 1. Suppose that $u \in V(G) \setminus S_G$. Since S_G is a secure fair dominating set of noncomplete graph G, S_G is a fair dominating set and for each $u \in V(G) \setminus S_G$, there exists $u^{'} \in S_G$ such that $uu^{'} \in E(G)$ and $(S_G \setminus \{u^{'}\}) \cup \{u\}$ is a fair dominating set of G. Since $|N_G(u) \cap S_G| = |N_G(u^{'}) \cap [(S_G \setminus \{u^{'}\}) \cup \{u\}]|$, it follows that

$|N_{G+H}(u) \cap S| = |[N_G(u) \cup V(H)] \cap (S_G \cup S_H)|$
$= |[N_G(u) \cap (S_G \cup S_H)] \cup [V(H) \cap (S_G \cup S_H)]|$
$= |[N_G(u) \cap S_G] \cup S_H|$
$= |N_G(u) \cap S_G| + |S_H|$
$= |N_G(u^{'}) \cap [(S_G \setminus \{u^{'}\}) \cup \{u\}]| + |S_H|$
$= |(N_G(u^{'}) \cap [(S_G \setminus \{u^{'}\}) \cup \{u\}]) \cup S_H|$
$= |(N_G(u^{'}) \cup S_H) \cap ([(S_G \setminus \{u^{'}\}) \cup \{u\}] \cup S_H)|$
$= |[N_{G+H}(u^{'})] \cap [\left((S_G \cup S_H) \setminus \{u^{'}\}\right) \cup \{u\}]|$
$= |[N_{G+H}(u^{'})] \cap [(S \setminus \{u^{'}\}) \cup \{u\}]|$

Hence, $|N_{G+H}(u) \cap S| = |N_{G+H}(u^{'}) \cap S'|$ for every $u \in V(G) \setminus S_G \subset V(G+H) \setminus S$ and for some $u^{'} \in S_G$, that is, $S^{'} = (S \setminus \{u^{'}\}) \cup \{u\}$ is a fair dominating set of $G+H$. Since S and S' are fair dominating sets of $G+H$, it follows that S is a secure fair dominating set of $G+H$.

Case 2. Suppose that $u \in V(H) \setminus S_H$. Since S_H is a secure fair dominating set of noncomplete graph H, S_H is a fair dominating set and for each $u \in V(H) \setminus S_H$, there exists $u^{'} \in S_H$ such that $uu^{'} \in E(H)$ and $(S_H \setminus \{u^{'}\}) \cup \{u\}$ is a fair dominating set of H. Since $|N_H(u) \cap S_H| = |N_H(u^{'}) \cap [(S_H \setminus \{u^{'}\}) \cup \{u\}]|$, it follows that

$|N_{G+H}(u) \cap S| = |[V(G) \cup N_H(u)] \cap (S_G \cup S_H)|$
$= |[V(G) \cap (S_G \cup S_H)] \cup [N_H(u) \cap (S_G \cup S_H)]|$
$= |S_G \cup [N_H(u) \cap S_H]|$
$= |S_G| + |[N_H(u) \cap S_H]|$
$= |S_G| + |N_H(u^{'}) \cap [(S_H \setminus \{u^{'}\}) \cup \{u\}]|$
$= |S_G \cup (|N_H(u^{'}) \cap [(S_H \setminus \{u^{'}\}) \cup \{u\}])|$
$= |\left(S_G \cup N_H(u^{'})\right) \cap (S_G \cup [(S_H \setminus \{u^{'}\}) \cup \{u\}])|$
$= |N_{G+H}(u^{'}) \cap [\left((S_G \cup S_H) \setminus \{u^{'}\}\right) \cup \{u\}]|$
$= |N_{G+H}(u^{'}) \cap [(S \setminus \{u^{'}\}) \cup \{u\}]|$

Hence, $|N_{G+H}(u) \cap S| = |N_{G+H}(u^{'}) \cap S'|$ for every $u \in V(H) \setminus S_H \subset V(G+H) \setminus S$ and for some $u^{'} \in S_H$, that is, $S^{'} = (S \setminus \{u^{'}\}) \cup \{u\}$ is a fair dominating set of $G+H$. Since S and S' are fair dominating sets of $G+H$, it follows that S is a secure fair dominating set of $G+H$. ■

Lemma 2.5 Let G and H be nontrivial connected graphs. Then a nonempty subset $S \subset V(G+H)$ is a secure fair dominating set of $G+H$, if S_G and S_H are nonempty subsets of complete graphs G and Hrespectively.

Proof: Let G and H be complete graphs. Clearly $G+H$ is a complete graph. Let $S \subset V(G+H)$. Then S is a secure fair dominating set of a complete graph $G+H$, by Remark 2.1. ■

Theorem 2.6 Let G and H be nontrivial connected graphs. Then a nonempty $S \subset V(G+H)$ is a secure fair dominating set of $G+H$, if one of the following conditions is satisfied.

$(i) S = V(G) \cup S_H$ where S_H is a secure fair dominating set of a noncomplete graph H.
$(ii) S = S_G \cup V(H)$ where S_G is a secure fair dominating set of a noncomplete graph G.
$(iii) S = S_G \cup S_H$ where
$a) S_G$ and S_H are secure fair dominating sets of noncomplete graphs G and H respectively, $|S_G| = |S_H|$, and for each $x \in V(G) \setminus S_G$ and $y \in V(H) \setminus S_H$, there exists $x' \in S_G$ such that

$$|N_G(x) \cap S_G| = |N_G(x') \cap [(S_G \setminus \{x'\}) \cup \{x\}]|$$

and there exists $y' \in S_H$ such that

$$|N_H(y) \cap S_H| = |N_H(y') \cap [(S_H \setminus \{y'\}) \cup \{y\}]|,$$

$b)$ or S_G and S_H are nonempty subsets of complete graphs G and Hrespectively.

Proof: Let G and H be nontrivial connected graphs.

Suppose that statement (i) is satisfied. Then a nonempty subset $S \subset V(G+H)$ is a secure fair dominating set of $G+H$, by Lemma 2.2.

Suppose that statement (ii) is satisfied. Then a nonempty subset $S \subset V(G+H)$ is a secure fair dominating set of $G+H$, by Lemma 2.3.

Suppose that statement $(iii)a$ is satisfied. Then a nonempty subset $S \subset V(G+H)$ is a secure fair dominating set of $G+H$, by Lemma 2.4.

Suppose that statement $(iii)b$ is satisfied. Then a nonempty subset $S \subset V(G+H)$ is a secure fair dominating set of $G+H$, by Lemma 2.5. This completes the proofs. ■

The following result is an immediate consequence of Theorem 2.6.

Corollary 2.7 Let G and H be nontrivial connected graphs of order m and order n respectively. Then

$$\gamma_{sfd}(G+H) = \begin{cases} 1 & if\ G\ and\ H\ are\ complete\ graphs \\ n & if\ G\ is\ a\ complete\ graph\ and\ H = P_1 + \overline{K}_{n-1} \\ m & if\ m = P_1 + \overline{K}_{m-1}\ and\ H\ is\ a\ complete\ graph \end{cases}$$

Proof: Let G and H be nontrivial connected graphs of order m and order n respectively. Consider the following cases.

Case 1. Suppose that G and H are complete graphs. Then $G+H$ is a complete graph and by Remark 2.1,$\gamma_{sfd}(G+H) = 1$.

Case 2. Suppose that G is a complete graph and $H = P_1 + \overline{K}_{n-1}$. Let $S = V(H)$ and $x, x' \in V(G+H) \setminus S = V(G)$. Then

$$\begin{aligned} |N_{G+H}(x) \cap S| &= |[N_G(x) \cup V(H)] \cap V(H)| \\ &= |[(V(G) \setminus \{x\}) \cup V(H)] \cap V(H)| \\ &= |V(H)| \\ &= |[(V(G) \setminus \{x'\}) \cup V(H)] \cap V(H)| \\ &= |N_{G+H}(x') \cap S|. \end{aligned}$$

Thus, $|N_{G+H}(x) \cap S| = |N_{G+H}(x^{'}) \cap S|$ for all $x, x^{'} \in V(G+H) \setminus S = V(G)$, that is, S is a fair dominating set of $G+H$.

Let $V(G) = \{x_1, x_2, \ldots, x_m\}$, $V(P_1) = \{y\}$ and $V(\overline{K}_{n-1}) = \{y_1, y_2, \ldots, y_{n-1}\}$. Then $V(G+H) = V(G) \cup V(H) = \{x_1, x_2, \ldots, x_m, y, y_1, y_2, \ldots, y_{n-1}\}$. Let $S = V(H) = \{y, y_1, y_2, \ldots, y_{n-1}\}$.For every $x \in V(G+H) \setminus S = V(G)$, there exists $y \in S$ such that $xy \in E(G+H)$ and $S^{'} = (S \setminus \{y\}) \cup \{x\} = \{y_1, y_2, \ldots, y_{n-1}, x\}$. To show that S' is a fair dominating set of $G+H$, suppose that $x^{'}, y \in V(G+H) \setminus S^{'} = (V(G) \setminus \{x\}) \cup \{y\}$. Then

$$\begin{aligned}
&|N_{G+H}(y) \cap S'| = |[V(G) \cup N_H(y)] \cap S'| \\
&= |[\{x_1, x_2, \ldots, x_m\} \cup \{y_1, y_2, \ldots, y_{n-1}\}] \cap \{y_1, y_2, \ldots, y_{n-1}, x\}| \\
&= |\{y_1, y_2, \ldots, y_{n-1}, x\}| \\
&= |[(\{x_1, x_2, \ldots, x_m\} \setminus \{x^{'}\}) \cup \{y, y_1, y_2, \ldots, y_{n-1}\}] \cap \{y_1, y_2, \ldots, y_{n-1}, x\}| \\
&= |[N_G(x^{'}) \cup V(H))] \cap S'| \\
&= |N_{G+H}(x^{'}) \cap S'|.
\end{aligned}$$

Thus, for all $x^{'}, y \in V(G+H) \setminus S^{'} = (V(G) \setminus \{x\}) \cup \{y\}$, $|N_{G+H}(x) \cap S'| = |N_{G+H}(y) \cap S'|$ implies that S' is a fair dominating set of $G+H$.

Since S is a fair dominating set of $G+H$ and for every $x \in V(G+H) \setminus S$ there exists $y \in S$ such that $xy \in E(G+H)$ and $S^{'} = (S \setminus \{y\}) \cup \{x\}$ is a fair dominating set of $G+H$, it follows that S is a secure fair dominating set of $G+H$.

Suppose that $S = \{y, y_1, y_2, \ldots, y_{n-1}\}$ is not a minimum secure fair dominating set of $G+H$.Then there exists $y^{'} \in S$ such that $S \setminus \{y'\}$ is a secure fair dominating set of $G+H$. However, $S \setminus \{y\}$ is not a dominating set of $G+H$, and for any $y^{'} \in S \setminus \{y\}$,$S \setminus y'$ is not a fair dominating set of $G+H$. Further, if $S \setminus \neq V(H)$, then S is either not fair or secure dominating set of $G+H$. Hence, $S = V(H) = \{y, y_1, y_2, \ldots, y_{n-1}\}$ must be a minimum dominating set of $G+H$. Therefore, $\gamma_{sfd}(G) = |S| = n$.

Case 3. Suppose that $G = P_1 + \overline{K}_{m-1}$ and H is a complete graph. Let $S = V(G)$ and $y, y^{'} \in V(G+H) \setminus S = V(H)$. Then

$$\begin{aligned}
&|N_{G+H}(y) \cap S| = |[V(G) \cup N_H(y)] \cap V(G)| \\
&= |[V(G) \cup (V(H) \setminus \{y\})] \cap V(G)| \\
&= |V(G)| \\
&= |[V(G) \cup (V(H) \setminus \{y^{'}\})] \cap V(G)| \\
&= |N_{G+H}(y^{'}) \cap S|.
\end{aligned}$$

Thus, $|N_{G+H}(y) \cap S| = |N_{G+H}(y^{'}) \cap S|$ for all $y, y^{'} \in V(G+H) \setminus S = V(H)$, that is, S is a fair dominating set of $G+H$.

Let $V(G) = \{x, x_1, x_2, \ldots, x_{m-1}\}$, $V(P_1) = \{x\}$ and $V(\overline{K}_{m-1}) = \{x_1, x_2, \ldots, x_{n-1}\}$. Then $V(G+H) = V(G) \cup V(H) = \{x, x_1, x_2, \ldots, x_{m-1}, y_1, y_2, \ldots, y_n\}$. Let $S = V(G) = \{x, x_1, x_2, \ldots, x_{m-1}\}$.For every $y \in V(G+H) \setminus S = V(H)$, there exists $x \in S$ such that $yx \in E(G+H)$ and $S^{'} = (S \setminus \{x\}) \cup \{y\} = \{x_1, x_2, \ldots, x_{m-1}, y\}$. To show that S' is a fair dominating set of $G+H$, suppose that $x, y^{'} \in V(G+H) \setminus S^{'} = (V(H) \setminus \{y\}) \cup \{x\}$. Then

$$\begin{aligned}
&|N_{G+H}(y^{'}) \cap S'| = |[V(G) \cup N_H(y^{'})] \cap S'| \\
&= |[\{x, x_1, x_2, \ldots, x_{m-1}\} \cup (\{y_1, y_2, \ldots, y_n\} \setminus \{y^{'}\})] \cap \{x_1, x_2, \ldots, x_{m-1}, y\}| \\
&= |\{x_1, x_2, \ldots, x_{m-1}, y\}| \\
&= |[(\{x_1, x_2, \ldots, x_m\}) \cup \{y_1, y_2, \ldots, y_n\}] \cap \{x_1, x_2, \ldots, x_{m-1}, y\}| \\
&= |[N_G(x) \cup V(H))] \cap S'| \\
&= |N_{G+H}(x) \cap S'|.
\end{aligned}$$

Thus, for all $x, y^{'} \in V(G+H) \setminus S^{'} = (V(H) \setminus \{y\}) \cup \{x\}$, $|N_{G+H}(y^{'}) \cap S'| = |N_{G+H}(x) \cap S'|$ implies that S' is a fair dominating set of $G+H$.

Since S is a fair dominating set of $G+H$ and for every $x \in V(G+H)\backslash S$ there exists $y \in S$ such that $xy \in E(G+H)$ and $S^{'} = (S \setminus \{y\}) \cup \{x\}$ is a fair dominating set of $G+H$, it follows that S is a secure fair dominating set of $G+H$.

Suppose that $S = \{x, x_1, x_2, \ldots, x_{m-1}\}$ is not a minimum secure fair dominating set of $G+H$.Then there exists $x^{'} \in S$ such that $S \setminus \{x'\}$ is a secure fair dominating set of $G+H$. However, $S \setminus \{x\}$ is not a dominating set of $G+H$, and for any $x^{'} \in S \setminus \{x\}$, $S \setminus x'$ is not a fair dominating set of $G+H$. Further, if $S \neq V(G)$, then S is either not a fair or secure dominating set of $G+H$. Hence, $S = V(G) = \{x, x_1, x_2, \ldots, x_{m-1}\}$ must be a minimum dominating set of $G+H$. Therefore, $\gamma_{sfd}(G) = |S| = m$. This completes the proofs of the Corollary.■

III. CONCLUSION

In this paper, we present a binary operation of the secure fair domination in graphs - the join of two connected nontrivial graphs. First, we establish some results through Lemmas. Next, we consolidate the Lemmas to prove a Theorem. Lastly, we give the immediate consequence of the Theorem, the Corollary. This study will pave a way to new researches such as the other binary operations of two connected graphs - the corona, the Cartesian product, etc. Other parameters relating the secure fair domination in graphs may also be explored. Finally, the characterization of a secure fair domination in graphs is a challenging extension of this study.

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