

Outer-clique Weakly Convex Domination in Graphs

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Abstract: Let G be a connected simple graph. A subset S of $V(G)$ is a dominating set of G if for every $v \in V(G) \setminus S$, there exists $x \in S$ such that $xv \in E(G)$. A subset C of $V(G)$ is called a weakly convex set of G if for every two vertices $u, v \in C$, there exists a $u - v$ geodesic whose vertices belong to C . A dominating set of S which is also weakly convex is called a weakly convex dominating set of G . A weakly convex dominating set S is an outer-clique weakly convex dominating set of G if the graph $\langle V(G) \setminus S \rangle$ induced by $V(G) \setminus S$ is complete. The minimum cardinality of an outer-clique weakly convex dominating set of G , denoted by $\tilde{\gamma}_{cwc}(G)$, is called the outer-clique weakly convex domination number of G . In this paper, we initiate the study of the concept and provide the outer-clique weakly convex domination numbers in some special graphs. Further, we show the sufficient conditions of an outer-clique weakly convex dominating set in the join of two nontrivial connected graphs.

Index Terms: dominating, weakly convex, outer-clique, outer-clique weakly convex, join

I. Introduction

In 1958, the concept of domination was first introduced by C. Berge in his book *Théorie des Graphes et Ses Applications* [1]. It was in the 1962 translated version of this book *The Theory of Graphs and Its Applications* where the concept of "coefficient of external stability" which is now considered as domination number was formally defined by Berge [2]. By 1962, the terms "domination" and "dominating set" were first used by Ø. Ore in his book *Theory of Graphs*[3]. In 1977, domination in graphs became an area of study following the article *Towards a Theory of Domination in Graphs* published by E. J. Cockayne and S. T. Hedetniemi. Cockayne and Hedetniemi also introduced the accepted notation $\gamma(G)$ for the domination number of graph G [4]. Some studies on domination in graphs can be found in [5], [6], [7], [8], [9], [10], [11].

Let G be a connected simple graph. A subset S of $V(G)$ is a dominating set of G if for every vertex $v \in V(G) \setminus S$, there exists $u \in S$ such that $uv \in E(G)$. The domination number $\gamma(G)$ of G is the cardinality of a minimum dominating set in G . A graph G is connected if every two vertices of G are connected, that is, if G contains a $u - v$ path for every pair u, v of vertices of G . A graph G that is not connected is called disconnected. The distance between u and v is the smallest length of any $u - v$ path in G and is denoted by $d_G(u, v)$, or simply $d(u, v)$ if the graph G under consideration is clear. A $u - v$ path of length $d_G(u, v)$ is called a $u - v$ geodesic [12].

A subset S of $V(G)$ is called a weakly convex set in G if for every two vertices $u, v \in V(G)$, there exists a $u - v$ geodesic whose vertices belong to S . A set $S \subseteq V(G)$ is a weakly convex dominating set if S is a weakly convex and a dominating set. The weakly convex domination number $\gamma_{wcon}(G)$ of a graph G represents the minimum cardinality of a weakly convex dominating set. This concept was introduced by Jerzy Topp in 2002 and were formally established and explored by M. Lemańska in 2004 [13]. Some studies of weakly convex domination in graphs can be found in [14], [15], [16], [17].

Another type of domination in graphs is the outer-clique domination in graphs. Introduced by Ravina et.al. in 2018, an outer-clique domination in graphs is motivated by the concepts of outer-connected and clique domination in graphs [18]. A dominating set $S \subseteq V(G)$ is an outer-connected dominating set of G when either $S = V(G)$ or $\langle V(G) \setminus S \rangle$ is connected [19]. A dominating set $S \subseteq V(G)$ is a clique dominating set if the induced subgraph $\langle S \rangle$ is complete [20]. As defined in [21], a set $S \subseteq V(G)$ is an outer-clique domination in graphs if every vertex $v \in V(G) \setminus S$ is adjacent to some vertex $u \in S$ and the subgraph induced by $V(G) \setminus S$ is clique. Results of outer-clique domination in graphs under binary operations like corona and Cartesian product was also explored in [18].

Motivated by the concepts of weakly convex domination and outer-clique domination in graphs, this paper introduces a new type of domination in graphs: outer-clique weakly convex domination. A subset S of $V(G)$ is called an outer-clique weakly convex dominating set of G if S is a weakly convex dominating set of G and the subgraph $\langle V(G) \setminus S \rangle$ induced by $V(G) \setminus S$ is complete. The minimum cardinality of an outer-clique weakly convex dominating set of G , denoted by $\tilde{\gamma}_{cwc}(G)$, is called the outer-clique weakly convex domination number of

G . In this paper, we will study the different outer-clique weakly convex domination numbers in some special graphs such as path, cycle, complete graph, wheel, fan, and complete bipartite. Furthermore, we show the properties of an outer-clique weakly dominating set in the join of the nontrivial connected graphs.

II. Results

The following definitions will be needed throughout the study.

Definition 2.1 A simple path $P_n = [v_1, v_2, \dots, v_n]$ is a special graph of order $n \geq 1$ with $V(P_n) = \{v_1, v_2, \dots, v_n\}$ and $E(P_n) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n\}$.

Theorem 2.2 If $G = P_n$ then $\tilde{\gamma}_{cwc}(G) = n - 1$ for all $n \geq 2$.

Proof. Let $G = P_n = [v_1, v_2, \dots, v_n]$, for all integers $n \geq 2$. Let $S = \{v_1, v_2, \dots, v_{n-1}\}$. Then S is a weakly convex dominating set of G , while $\langle V(G) \setminus S \rangle = K_1$. Hence, S is an outer-clique weakly convex dominating set of G . For any $x \in S$, note that either $S \setminus \{x\}$ is not a dominating set or $\langle V(G) \setminus (S \setminus \{x\}) \rangle$ is not complete. Thus, S is a minimum outer-clique weakly convex dominating set of G . Therefore, $\tilde{\gamma}_{cwc}(G) = |S| = n - 1$, for all $n \geq 2$. \square

Definition 2.3 The cycle $C_n = [v_1v_2, v_2v_3, \dots, v_nv_1]$ is a special graph with $V(C_n) = \{v_1, v_2, \dots, v_n\}$ and $E(C_n) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1\}$, where $n \geq 3$.

Theorem 2.4 If $G = C_n$ then

$$\tilde{\gamma}_{cwc}(G) = \begin{cases} n - 2, & \text{for all } 3 \leq n \leq 6 \\ \text{none}, & \text{if } n \geq 7 \end{cases}$$

Proof. Let $G = C_n = [v_1, v_2, \dots, v_n, v_1]$ for all $n \geq 3$.

Case 1. Consider $3 \leq n \leq 6$. Let $S = \{v_2, v_3, \dots, v_{n-1}\}$. Then S is a weakly convex dominating set of G and $\langle V(G) \setminus S \rangle = K_2$. Thus, S is an outer-clique weakly convex dominating set of G . Note that if $x \in S$, then either $S \setminus \{x\}$ is not a dominating set of G or $\langle V(G) \setminus (S \setminus \{x\}) \rangle$ is not complete. Thus, S is a minimum outer-clique weakly convex dominating set of G . Hence, $\tilde{\gamma}_{cwc}(G) = |S| = n - 2$, for all $3 \leq n \leq 6$.

Case 2. Consider $n \geq 7$. Let $S = \{v_2, v_3, \dots, v_{n-1}\}$. Then S is a dominating set of G . Now, for any $x \in S$, notice that $S \setminus \{x\}$ is either not a dominating set of G or not connected. In addition, note that for any $n \geq 7$, $d(v_2, v_{n-1}) = 3$ and $I(v_2, v_{n-1}) = \{v_2, v_1, v_n, v_{n-1}\}$ is the set of all vertices that lie on the $v_2 - v_{n-1}$ geodesic. Since $v_1, v_n \in I(v_2, v_{n-1})$ but $v_1, v_n \notin S$, then S is not a weakly convex dominating set of G and is not an outer-clique weakly convex dominating set of G . Similarly, if $S = \{v_2, v_3, \dots, v_{n-1}, v_n\}$, then S is also not a weakly convex dominating set of G . Thus, there is no outer-clique weakly convex dominating set of G for $n \geq 7$, and so, $\tilde{\gamma}_{cwc}(G)$ is none. \square

Definition 2.5 The complete graph K_n is a special graph of order n , where every pair of vertices is adjacent.

By observation, the following remark gives the outer-clique weakly convex domination of the complete graph K_n , for all $n \geq 2$.

Remark 2.6 If $G = K_n$, then $\tilde{\gamma}_{cwc}(G) = 1$, for all $n \geq 2$.

Definition 2.7 The wheel W_n is a special graph of order $n + 1$ with $V(W_n) = \{v_0, v_1, v_2, \dots, v_n\}$ and $E(W_n) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1\} \cup \{v_0v_i : i = 1, 2, \dots, n\}$, where $n \geq 3$.

Theorem 2.8 Let $G = W_n = K_1 + C_n$, for all $n \geq 3$, then

$$\tilde{\gamma}_{cwc}(G) = \begin{cases} n - 2, & \text{if } 3 \leq n \leq 5 \\ n - 1, & \text{if } n \geq 6 \end{cases}$$

Proof. Let $G = W_n = K_1 + C_n$, for all $n \geq 3$.

Case 1. Consider $3 \leq n \leq 5$. Let $S = \{v_1, \dots, v_{n-2}\}$. Then S is a weakly convex dominating set of G and $\langle V(G) \setminus S \rangle = K_3$. Hence, S is also an outer-clique weakly convex dominating set of G . Now, for any $x \in S$, either

$S \setminus \{x\}$ is not a dominating set of G or $\langle V(G) \setminus (S \setminus \{x\}) \rangle$ is not complete. Thus, S is a minimum outer-clique weakly convex dominating set of G , and $\tilde{\gamma}_{cwc}(G) = |S| = n - 2$.

Case 2. Consider $n \geq 6$. Let $S = \{v_0, v_1, \dots, v_{n-2}\}$. Then S is a weakly convex dominating set of G . Note that $\langle V(G) \setminus S \rangle = K_2$, then S is an outer-clique weakly convex dominating set of G . Since for any $x \in S$, either S is not a dominating set of G or $\langle V(G) \setminus (S \setminus \{x\}) \rangle$ is not complete, then S is a minimum outer-clique weakly convex dominating set of G . Thus, $\tilde{\gamma}_{cwc}(G) = |S| = n - 1$. \square

Definition 2.9 A simple fan $F_n = K_1 + P_n$ is a special graph of order $n + 1 \geq 2$ with $V(F_n) = \{v_0, v_1, v_2, \dots, v_n\}$ and $E(F_n) = \{v_1 v_2, v_2 v_3, \dots, v_{n-1} v_n\} \cup \{v_0 v_i : i = 1, 2, \dots, n\}$.

Theorem 2.10 If $G = F_n$, where $F_n = K_1 + P_n$, for all $n \geq 1$, then

$$\tilde{\gamma}_{cwc}(G) = \begin{cases} 1, & \text{if } n = 1 \\ n - 1, & \text{if } n \geq 2 \end{cases}$$

Proof. Let $G = F_n = K_1 + P_n$, for all $n \geq 1$.

Case 1. If $n=1$, then $G = F_1 = K_2$. By Remark 2.6, $\tilde{\gamma}_{cwc}(G) = 1$.

Case 2. Consider $n \geq 2$. Let $S = \{v_0, v_1, \dots, v_{n-2}\}$. Note that $d(u, v) \leq 2$, for all $u, v \in V(G)$. Now, since $v_0 \in S$, it follows that a $u - v$ geodesic exists, where $I(u, v) \subseteq S$, for all $u, v \in S$. Hence, S is a weakly convex dominating set of G . Since $\langle V(G) \setminus S \rangle = K_2$, then S is an outer-clique weakly convex dominating set of G . In addition, since for any $x \in S$, either $S \setminus \{x\}$ is not a dominating set of G or $\langle V(G) \setminus (S \setminus \{x\}) \rangle$ is not complete, then S is a minimum outer-clique weakly convex dominating set of G . Thus $\tilde{\gamma}_{cwc}(G) = |S| = 1 + (n - 2) = n - 1$. \square

Definition 2.11 A simple complete bipartite $K_{m,n} = \overline{K_m} + \overline{K_n}$ is a special graph of order $m + n$ such that $m = |V(\overline{K_m})|$ and $n = |V(\overline{K_n})|$. A complete bipartite has a vertex set which can be partitioned into V_1 and V_2 such that every edge joins a vertex in V_1 with a vertex in V_2 , and every vertex in V_1 is adjacent with every vertex in V_2 .

Theorem 2.12 If $G = K_{m,n}$, then

$$\tilde{\gamma}_{cwc}(G) = \begin{cases} n, & \text{if } (m = 1 \text{ and } n \geq 1) \text{ or } (m = 2 \text{ and } n \neq 1) \\ m, & \text{if } (m \geq 1 \text{ and } n = 1) \text{ or } (m \neq 1 \text{ and } n = 2) \\ m + n - 2, & \text{if } m \geq 3 \text{ and } n \geq 3 \end{cases}$$

Proof. Let $G = K_{m,n} = \overline{K_m} + \overline{K_n}$, for all $n \geq 1$. Let $u_i \in V(\overline{K_m})$ and $v_j \in V(\overline{K_n})$. Note that $d(u, v) \leq 2$, for all $u, v \in V(G)$.

Case 1. Consider $m = 1$ and $n \geq 1$. Let $S = \{u_1, v_1, v_2, \dots, v_{n-1}\}$. Note that $d(u, v) \leq 2$, for all $u, v \in V(G)$. Since $u_1 \in S$, and $u_1 v_j \in E(G)$, for all j , then $d_S(v_p, v_q) \leq 2$, for all $p, q = 1, 2, \dots, n - 1$. Thus, S is a weakly convex dominating set of G . In addition, $\langle V(G) \setminus S \rangle = K_1$. Hence, S is an outer-clique weakly convex dominating set of G . For any $x \in S$, then either $S \setminus \{x\}$ is not a dominating set of G or $\langle V(G) \setminus (S \setminus \{x\}) \rangle$ is not complete. Hence, S is a minimum outer-clique weakly convex dominating set of G . Thus, $\tilde{\gamma}_{cwc}(G) = |S| = 1 + (n - 1) = n$.

Case 2. Consider $m = 2$ and $n \neq 1$. Let $S = \{u_1, v_1, v_2, \dots, v_{n-1}\}$. Since $u_1 \in S$, then $d_S(v_p, v_q) \leq 2$, for all $p, q = 1, 2, \dots, n - 1$. Thus, S is a weakly convex dominating set of G . Moreover, $\langle V(G) \setminus S \rangle = K_2$. Now, for any $x \in S$, either $S \setminus \{x\}$ is not a dominating set of G or $\langle V(G) \setminus (S \setminus \{x\}) \rangle$ is not complete. Hence, S is a minimum outer-clique weakly convex dominating set of G . Thus, $\tilde{\gamma}_{cwc}(G) = |S| = 1 + (n - 1) = n$.

Case 3. Now, consider $n = 1$ and $m \geq 1$. Then by Case 1, $S = \{u_1, u_2, \dots, u_{m-1}, v_1\}$ is a minimum outer-clique weakly convex dominating set of G . That is, $\tilde{\gamma}_{cwc}(G) = |S| = (m - 1) + 1 = m$.

Case 4. Consider $m \neq 1$ and $n = 2$. Then by Case 2, $S = \{u_1, u_2, \dots, u_{m-1}, v_1\}$ is a minimum outer-clique weakly convex dominating set of G with $\tilde{\gamma}_{cwc}(G) = |S| = (m - 1) + 1 = m$.

Case 5. Consider $m \geq 3$ and $n \geq 3$. Let $S = \{u_1, u_2, \dots, u_{m-1}, v_1, v_2, \dots, v_{n-1}\}$. Note that there exists $u \in V(\overline{K_m})$ and $v \in V(\overline{K_n})$, such that $u, v \in S$. Now, since $G = K_{m,n} = \overline{K_m} + \overline{K_n}$ is a join of graphs, then $d_S(u_p, v_q) \leq 2$, for all $u_p, v_q \in S$. That is, S is a weakly convex dominating set of G . On the other hand, $\langle V(G) \setminus S \rangle = K_2$. Hence, S is an outer-clique weakly convex dominating set of G . In particular, for any $x \in S$, either $S \setminus \{x\}$ is not a dominating set of G or $\langle V(G) \setminus (S \setminus \{x\}) \rangle$ is not complete. This implies that S is a minimum outer-clique weakly convex dominating set of G , and so, $\tilde{\gamma}_{cwc}(G) = |S| = (m-1) + (n-1) = m+n-2$. \square

Definition 2.13 The join of $G + H$ of two graphs G and H is the graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Theorem 2.14 Let G and H be nontrivial connected graphs. The subset $S \subseteq V(G + H)$ is an outer-clique weakly convex dominating set of $G + H$ if $\langle V(G + H) \setminus S \rangle$ is a complete subgraph of $G + H$, where $V(G + H) \setminus S \neq \emptyset$ and one of the following conditions is satisfied.

- (i) $S = S_G \cup \{v\}$ such that $S_G \subseteq V(G)$, $v \in V(H)$.
- (ii) $S = \{u\} \cup S_H$ such that $S_H \subseteq V(H)$, $u \in V(G)$.
- (iii) $S = S_G \cup S_H$.

Proof. Let G and H be nontrivial connected graphs. Let $S \subseteq V(G + H)$. Suppose that $\langle V(G + H) \setminus S \rangle$ is a complete subgraph of $G + H$, where $V(G + H) \setminus S \neq \emptyset$. Let $u_i \in V(G)$ and $v_j \in V(H)$. Note that $d_{G+H}(x, y) \leq 2$, for all $x, y \in V(G + H)$.

Case 1. Let $S = S_G \cup \{v\}$ such that $S_G \subseteq V(G)$ and $v \in V(H)$. Take any $u_m \in S_G$. Then $u_m v_p \in E(G + H)$, for all $v_p \in V(H)$. Hence, S is a dominating set of $G + H$. In addition, since $v \in S$, then $d_S(a, b) \leq 2$, for all $a, b \in S$ and $I(x, y) \subseteq S$. Thus, S is a weakly convex dominating set of $G + H$. By assumption, since $\langle V(G + H) \setminus S \rangle$ is complete, then S is an outer-clique weakly convex dominating set of $G + H$.

Case 2. Let $S = S_H \cup \{u\}$ such that $S_H \subseteq V(H)$ and $u \in V(G)$. Take any $v_p \in S_H$. Then v_p dominates all vertices in $V(G)$. Since $u \in S$, then u dominates all vertices in $V(H)$. Hence, S is a dominating set of $G + H$. In particular, since $u \in S$, then $d_S(a, b) \leq 2$, and $I(a, b) \subseteq S$, for all $a, b \in S$, implies that S is also a weakly convex dominating set of G . Note that $\langle V(G + H) \setminus S \rangle$ is complete, then S is an outer-clique weakly convex dominating set of $G + H$.

Case 3. Let $S = S_G \cup S_H$. Clearly, S is a dominating set of $G + H$ and $d_S(u, v) \leq 2$, for all $u, v \in S$. That is, $I(x, y) \subseteq S$, for all $x, y \in S$. Thus, S is a weakly convex dominating set of $G + H$. Since $\langle V(G + H) \setminus S \rangle$ is complete as given, it follows that S is an outer-clique weakly convex dominating set of $G + H$. \square

III. Conclusion

In this paper, a new type of domination was introduced, that is, an outer-clique weakly convex domination in graphs. The outer-clique weakly convex domination numbers in various special graphs such as path, cycle, complete graph, wheel, fan, and complete bipartite were provided. In addition, sufficient conditions as to when there exists an outer-clique weakly convex dominating set in a join of nontrivial connected graphs was also provided. Full characterization of this parameter under join of graphs can also be explored together with its properties under the said binary operation. This study can be extended further for future researches on the properties of outer-clique weakly convex domination number of graphs under the binary operations such as corona, and Cartesian product.

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