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Disjoint Connected Domination in Graphs

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Abstract: Let G be a connected simple graph. A subset S of V(G) is a dominating set of G if for every $v \in V(G) \setminus S$, there exists $x \in S$ such that $xv \in E(G)$. A connected dominating set C of a graph G is a dominating set of G such that the subgraph induced by the vertices of C in G is connected. Let C be a minimum connected dominating set of G. The connected dominating set $S \subseteq V(G) \setminus C$is called an inverse connected dominating set of Gwith respect to C. A disjoint connected dominating set of G is the set $D = C \cup S \subseteq V(G)$. The minimum cardinality of a disjoint connected dominating set of G, denoted by $\gamma\gamma_c(G)$, is called the disjoint connected domination number of G. In this paper, we initiate the study of the concept and give the domination number of special graphs. Further, we show the characterization of the disjoint connected dominating set in the join of two nontrivial connected graphs.

Index Terms: dominating, connected, disjoint, disjoint connected, join

I. Introduction

The concept of domination in graphs was first introduced by C. Berge in 1958 in his book *Théorie des Graphes et Ses Applications* [1]. In the 1962 translated version, *The Theory of Graphs and Its Applications*, the concept of the *coefficient of external stability*, now known as the domination number, was formally defined [2]. In the same year, the terms *domination* and *dominating set* were first used by Ø. Ore in *Theory of Graphs* [3]. The study of domination gained further development following the influential work *Towards a Theory of Domination in Graphs* by E. J. Cockayne and S. T. Hedetniemi in 1977, where the standard notation$\gamma(G)$ for the domination number was introduced [4]. Since then, numerous studies have contributed to the development of domination theory and its various extensions and applications, including total domination, connected domination, and several recent variants of domination in graphs [5],[6],[7],[8],[9],[10],[11].

To establish the necessary background for this study, we recall some basic concepts in graph theory from Chartrand and Zhang [12]. Let $G = (V(G), E(G))$be a simple graph, where $V(G)$is a non-empty finite set of vertices and $E(G)$is a set of edges connecting pairs of distinct vertices. Two vertices are said to be adjacent if they are connected by an edge. The neighborhood of a vertex *v*, denoted by $N(v)$, is the set of all vertices adjacent to *v*, and the degree of *v* is the number of vertices in its neighborhood. A graph *G* is connected if there exists a path between every pair of distinct vertices; otherwise, it is disconnected. For a subset$S \subseteq V(G)$, the induced subgraph $\langle S \rangle$ is the graph whose vertex set is S and whose edge set consists of all edges of G joining vertices in S. Let G be a connected simple graph. A subset $S \subseteq V(G)$ is called a *dominating set* of G if for every vertex $v \in V(G) \setminus S$, there exists a vertex $u \in S$ such that $uv \in E(G)$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G[12].

One important variation is *connected domination*, which incorporates connectivity into the structure of dominating sets. A subset $S \subseteq V(G)$ is called a *connected dominating set* if it is a dominating set and the induced subgraph $\langle S \rangle$ is connected. The connected domination number, denoted by $\gamma_c(G)$, is the minimum cardinality of a connected dominating set in G. This concept has important applications in communication networks, particularly in the design of virtual backbones where connectivity among selected vertices is essential [13],[14],[15].

The concept of inverse domination, which studies relationships between dominating sets and their complements, was introduced by V. R. Kulli and S. C. Sigarkanti in 1991 [16]. This idea, together with related work on redundancy in domination structures, motivates the study of alternative dominating configurations in graphs. Closely related to this is the concept of *disjoint domination*, where multiple dominating sets are required to be pairwise disjoint. A graph G is said to have disjoint dominating sets if there exist dominating sets $S_1, S_2 \subseteq V(G)$such that $S_1 \cap S_2 = \emptyset$. This concept is closely associated with redundancy and fault tolerance in network systems, allowing alternative domination structures in case of failures. Various studies on disjoint domination and its variants can be found in [17],[18],[19],[20],[21],[22].

Motivated by connected domination and disjoint domination, we consider the concept of *disjoint connected domination in graphs*. Two disjoint sets $C, S \subseteq V(G)$ are considered such that both induce connected dominating structures. Let C be a minimum connected dominating set of G, and let $S \sqsubseteq V(G)\backslash C$ serve as an inverse connected dominating set with respect to C. The union $D = C \cup S$ is called a *disjoint connected dominating set* of G, and the minimum cardinality of such a set is denoted by $\gamma\gamma_c(G)$. In this paper, we investigate disjoint connected domination in various graph classes and examine its structural properties. We also characterize disjoint connected dominating sets in the join of two nontrivial connected graphs. Throughout this paper, all graphs are finite, simple, and connected.

II. Results

The following definitions will be needed throughout the study.

Definition 2.1: A path $P_n = a_1 a_2 \ldots a_n$ is a graph with vertex set $V(P_n) = \{a_1, a_2, \ldots, a_n\}$ and edge set $E(P_n) = \{a_1 a_{2,} a_2 a_3, \ldots, a_{n-1} a_n\}$.

Proposition 2.2: Let $G = P_n$ for all $n \geq 2$. Then

$$\gamma\gamma_c(G) = \begin{cases} 2, & if\ n = 2 \\ none, & if\ n \geq 3 \end{cases}$$

Proof: Let $P_n = [v_1, v_2, \ldots, v_n]$ be a path graph. Consider the case $n = 2$. Let $C = v_1$ and $S = v_2$.
Clearly, $C \cap S = \emptyset$. Moreover, each of C and S is a connected dominating set of P_2 since v_1 dominates v_2 and v_2 dominates v_1. Hence, $D = C \cup S = V(P_2)$ is a disjoint connected dominating set. Therefore, $\gamma\gamma_c(P_2) = 2$.
Now suppose that $n \geq 3$. Let C be a minimum connected dominating set of P_n. Since P_n is a path, every connected induced subgraph of P_n is itself a subpath consisting of consecutive vertices. In order for C to dominate the end vertex v_1, it is necessary that $v_2 \in C$. Similarly, to dominate v_n,it is necessary that $v_{n-1} \in C$. Because C must also be connected, it must contain every vertex lying on the unique path between v_2 and v_{n-1}. Hence, $C = v_2, v_3, \ldots, v_n$. Therefore, $V(P_n)\backslash C = \{v_1, v_n\}$.
To determine whether an inverse connected dominating set exists with respect to C, let $S \subseteq V(P_n)\backslash C$. Then the only possible choices for S are $\{v_1\}$, $\{v_n\}$, and $\{v_1, v_n\}$. If $S = \{v_1\}$, then v_n is not dominated by S. If $S = \{v_n\}$, then v_1 is not dominated by S. If $S = \{v_1, v_n\}$, then S is a dominating set, but the induced subgraph $\langle V(G)\backslash S \rangle$ is disconnected since $v_1 v_n \notin E(P_n)$ for $n \geq 3$. Thus, no subset of $V(P_n)\backslash C$ is a connected dominating set. It follows that P_n admits no inverse connected dominating set for every $n \geq 3$. Consequently, $\gamma\gamma_c(P_2)$ is none. ☐

Definition 2.3: The cycle $C_n = a_1 a_2 \ldots a_n a_1$ is the graph with $V(C_n) = \{a_1, a_2, \ldots, a_n\}$ and

$$E(P_n) = \{a_1 a_{2,} a_{2a_3}, \ldots, a_{n-1} a_n, a_n a_1\}.$$

Proposition 2.4: Let $G = C_n$ for all $n \geq 3$. Then,

$$\gamma\gamma_c(G) = \begin{cases} 2, & if\ n = 3 \\ 4, & if\ n = 4 \\ none, & if\ n \geq 5 \end{cases}$$

Proof: Let $C_n = [v_1, v_2, \ldots, v_n, v_1]$ be a cycle graph.

Case 1: $n = 3$. Since $C_3 \cong K_3$, every single vertex is a connected dominating set. Take $C = \{v_1\}$ and $S = v_2$. Clearly, $C \cap S = \emptyset$,and both C and S are connected dominating set of C_3. Hence, $D = C \cup S$ is a disjoint connected dominating set with $|D| = 2$. Therefore, $\gamma\gamma_c(C_3) = 2$.
Case 2: $n = 4$. Let $C = \{v_1, v_2\}$ and $S = \{v_3, v_4\}$. Clearly, $C \cap S = \emptyset$. The induced subgraphs $\langle V(G)\backslash C \rangle$ and $\langle V(G)\backslash S \rangle$ are both connected. Moreover, C dominates v_3 and v_4, while S dominates v_1 and v_2. Thus both C and S are connected dominating sets of G. Hence, $D = C \cup S$ is a disjoint connected dominating set G, and $|D| = 4$. Therefore, $\gamma\gamma_c(C_4) = 4$.
Case 3: $n \geq 5$. Let C be a minimum connected dominating set of C_n. For a cycle C_n, a minimum connected dominating set is obtained by removing any two adjacent vertices. Thus, without loss of generality, $C = \{v_2, v_3, \ldots, v_{n-1}\}$. Hence, $V(C_n)\backslash C = \{v_1, v_n\}$. Let $S \subseteq V(C_n)\backslash C$. Then the only possible choices are $S = \{v_1\}$, $S = \{v_n\}$ or $S = \{v_1, v_n\}$. If $S = \{v_1\}$, then vertices $v_3, \ldots, v_{n-1}$ are not dominated. If $S = \{v_n\}$, then vertices $v_2, \ldots, v_{n-2}$ are not dominated. If $S = \{v_1, v_n\}$, then S is connected since $v_1 v_n \in E(C_n)$, but it does not dominate all vertices of C_n for $n \geq 5$, since at least one vertex among $v_3, v_4, \ldots, v_{n-2}$ is not adjacent to either v_1 or v_n. Thus, no subset of $V(C_n)\backslash C$ is a connected dominating set for $n \geq 5$. Therefore, no inverse connected dominating set exists, and so $\gamma\gamma_c(C_n)$ is none. ☐

Definition 2.5: The complete graph K_n is the graph of order n where every pair of its vertices is adjacent.

Proposition 2.6: Let $G = K_n$ for all $n \geq 2$. Then $\gamma\gamma_c(G) = 2$.

***Proof*:** Let $G = K_n$ for $n \geq 2$. Since G is a complete graph, every vertex is adjacent to every other vertex. Hence, any single vertex forms a connected dominating set. Choose two distinct vertices $u, v \in V(K_n)$, and let $C = \{u\}$ and $S = \{v\}$. Clearly, $C \cap S = \emptyset$. Since every vertex of K_n is adjacent to both u and v, each of C and S dominates K_n. Moreover, the induced subgraphs $\langle V(G) \backslash C \rangle$ and $\langle V(G) \backslash S \rangle$ are trivially connected, since each consists of exactly one vertex. Thus, C and S are disjoint connected dominating sets of K_n. Therefore, $D = C \cup S$ is a disjoint connected dominating set with $|D| = 2$, it follows that $\gamma\gamma_c(G)$ =2.☐

Definition 2.7: A simple fan $F_n = K_1 + P_n$is a special graph of order $n + 1 \geq 2$with $V(F_n)$ =$\{a_{0,}a_1, a_2, \ldots, a_n\}$ and $E(F_n) = \{a_1a_2, a_2a_3, \ldots, a_{n-1}a_n, a_na_1\} \cup \{a_0a_i : i = 1,2, \ldots, n\}$.

Proposition 2.8: Let $G = F_n$ where $F_n = K_1 + P_n$ for all $n \geq 1$. Then,

$$\gamma\gamma_c(G) = \begin{cases} 2, & if \ n = 1 \ or \ n = 2 \\ n - 1, & if \ n \geq 3 \end{cases}$$

***Proof*:** Let $F_n = K_1 + P_n$, where u is the vertex of K_1 and $P_n = [v_1, v_2, \ldots, v_n]$.

***Case 1*:** $n = 1$. Then $F_1 \cong K_2$. By Proposition 2.6, $\gamma\gamma_c(F_1) = 2$.

***Case 2*:** $n = 2$. Let $V(F_2) = \{u, v_1, v_2\}$ and $F_2 \cong K_3$. Hence, any single vertex is connected dominating set. Take
$C = \{u\}$ and $S = \{v_1\}$. Then $C \cap S = \emptyset$,and both C and S are connected dominating sets. By Proposition 2.6, $\gamma\gamma_c(F_2) = 2$.

Case 3: $n \geq 3$. Let $C = \{u\}$. Since u is adjacent to every vertex of P_n, C is a connected dominating set of F_n. Now consider $S = \{v_2, \ldots, v_{n-1}\}$. Clearly, $C \cap S = \emptyset$.Moreover, $\langle V(G) \backslash S \rangle$ is connected since it induces the path $v_2v_3 \ldots v_{n-1}$. It remains to show that S is a dominating set. The vertex u is adjacent to every vertex in S, so u is dominated. Also, v_n is adjacent to $v_{n-1} \in S$. Every vertex in S dominates itself. Hence, S is a connected dominating set. Thus, $D = C \cup S = \{u, v_2, \ldots, v_{n-1}\}$ is a disjoint connected dominating set of F_n, and $|D| = |C| + |S| = 1 + (n - 2) = n - 1$. Since $C = \{u\}$ is a minimum connected dominating set and the smallest connected dominating subset of $V(F_n) \backslash C = \{v_2, v_3, \ldots, v_n\}$ that dominates v_n and remains connected is $\{v_2, v_3, \ldots, v_{n-1}\}$, it follows that the minimum size of the inverse connected dominating set of F_n is $n - 1$. Therefore, $\gamma\gamma_c(F_n) = n - 1$.

Definition 2.9: The wheel W_n is a special graph of order $n + 1$ with $V(W_n) = \{v_0, v_1, v_2, \ldots, v_n\}$ and $E(W_n) = \{a_1a_{2,}a_2a_3, \ldots, a_{n-1}a_n, a_na_1\} \cup \{a_0a_i : i = 1,2, \ldots, n\}$.

Proposition 2.10: Let $G = W_n$ where $W_n = K_1 + C_n$ for all $n \geq 3$. Then $\gamma\gamma_c(G) = n - 1$

***Proof*:** Let $W_n = K_1 + C_n$, where u is the vertex of K_1 and $C_n = [v_1, v_2, \ldots, v_{n-1}]$ is the cycle of order n, with $n \geq 3$. Take $C = \{u\}$. Since u is adjacent to every vertex of C_n, it follows that C is a connected dominating set of W_n. Now consider $S = \{v_1, v_2, \ldots, v_{n-2}\}$. Clearly, $C \cap S = \emptyset$. Since $v_1, v_2, \ldots, v_{n-2}$ are consecutive vertices on the cycle, the induced subgraph $\langle V(G) \backslash S \rangle$ is the path $v_1v_2 \ldots v_{n-2}$, which is connected. It remains to show that S is a dominating set of W_n. The central vertex u is adjacent to every vertex of S, so u is dominated. The remaining vertices $v_n, v_{n-1} \notin S$ are adjacent to v_1 and v_{n-2} repectively, and hence is also dominated by S. Therefore, S is a connected dominating set. Thus, S is an inverse connected dominating set with respect to C, and $|S| = n - 2$. Hence, $D = \{u, v_1, v_2, \ldots, v_{n-2}\}$, that is $|D| = n - 1$. To show minimality, suppose $|D| < n - 1$. Since $C = \{u\}$, any inverse connected dominating set S must satisfy $S \subseteq V(G) \backslash C = \{v_1, v_2, \ldots, v_n\}$. Hence, $S \leq n - 2$.

Assume, for contradiction, that there exists such a set S with $|S| \leq n - 3$ that is a connected dominating set of W_n. Then at least three vertices of the cycle C_n are not in S. Let these vertices be $v_i, v_j, v_k \notin S$, where they are taken in cyclic order. Since S must dominate all vertices, each of v_1, v_j, v_k must be adjacent to some vertex in S. However, removing at least three vertices from a cycle breaks it into at least two disjoint paths. Hence, the induced subgraph $\langle V(G)\backslash S\rangle$ cannot be connected, contradicting the assumption that S is a connected dominating set. Thus, any inverse connected dominating set with respect to C must have size at least $n-2$. Consequently, $|D| = |C| + |S| \geq 1 + (n-2) = n-1$. Since we have already constructed a set D with $|D| = n-1$, it follows that $\gamma\gamma_c(G) = n-1$. ☐

Definition 2.11: A simple complete bipartite graph $K_{m,n} = \overline{K_m} + \overline{K_n}$ is a special graph of order $m+n$, where $m = \left|V(\overline{K_m})\right|$ and $n = \left|V(\overline{K_n})\right|$. A complete bipartite graph has a vertex set that can be partitioned into two disjoint sets V_1 and V_2 such that every edge joins a vertex in V_1 with a vertex in V_2, and every vertex in V_1 is adjacent to every vertex in V_2.

Proposition 2.12: Let $G = K_{m,n}$ where $K_{m,n} = \overline{K_m} + \overline{K_n}$ for all $m \geq 2$ and $n \geq 2$. Then $\gamma\gamma_c(G)$ =4.

Proof: Let $G = K_{m,n} = \overline{K_m} + \overline{K_n}$ with bipartition $V(G) = A \cup B$ where $|A| = m$, $|B| = n$, and $2 \leq m \leq n$. Let C be a connected dominating set of G. Since, there are no edges within A and within B, a single vertex cannot dominate all vertices of G. Thus, C must contain at least one vertex from each partite set. Let $C = \{a, b\}$, where $a \in A$ and $b \in B$. Then a dominates all vertices in B, and b dominates all vertices in A. Moreover, since $ab \in E(G)$, the induced subgraph $\langle V(G)\backslash C\rangle$ is connected. Hence, C is a minimum connected dominating set of G. Now, $V(G)\backslash C = A\backslash\{a\} \cup B\backslash\{b\}$. To determine an inverse connected dominating set with respect to C, let $S \subseteq V(G)\backslash C$. Since there are no edges within A and within B, any connected subset must contain at least one vertex from each partite set. Consider $S = \{b'\} \cup \{a'\}$, where $a' \in A\backslash\{a\}$ and $b' \in B\backslash\{b\}$. Clearly, $C \cap S = \emptyset$. The induced subgraph $\langle V(G)\backslash C\rangle$ is connected since a' is adjacent to every vertex in $B\backslash\{b\}$ and b' is adjacent to every vertex in $A\backslash\{a\}$. Thus, S is a connected dominating set of G, and hence an inverse connected dominating set with respect to C. Therefore, $|S| = |\{a', b'\}| = 2$ and $|D| = |C \cup S| = |\{a, b, a', b'\}| = 4$. To establish minimality, suppose that $|D| < 4$. Since $D = C \cup S$ with $C \cap S = \emptyset$, this implies that either $|C| = 1$ or $|S| = 1$. However, $|C| = 1$ is impossible because no single vertex can dominate all vertices G. On the other hand, if $|S| = 1$, then $S = \{v\} \subseteq V(G)\backslash C$, but such a vertex cannot dominate all vertices in its own partite set, nd hence S is not a connected dominating set. This contradiction shows that no such set D with $|D| < 4$ exists. Therefore, $\gamma\gamma_c(G) = 4$. ☐

Definition 2.13: The join of two graphs $G + H$ is the graph with $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{uv: u \in V(G), v \in V(G)\}$.

Lemma 2.14: Let G and H be nontrivial connected graphs. A subset $D \subseteq V(G+H)$ is a disjoint connected dominating set of $G+H$ if $D = C \cup S_G$, where C is a minimum connected dominating set of G and also of $G+H$, and S_G is an inverse connected dominating set of G with respect to C.

***Proof*:** Let G and H be nontrivial connected graphs. Suppose that $D = C \cup S_G$, where C is a minimum connected dominating set of both G and $G+H$, and $S_G \subseteq V(G)\backslash C$ is an inverse connected dominating set of G with respect to C. We first show that D is a dominating set of $G+H$. Let $u \in V(G+H)\backslash D$. Then $V(G+H)\backslash D = (V(G) \cup V(H))\backslash(C \cup S_G)$. Since $C \cup S_G \subseteq V(G)$, it follows that $V(G+H)\backslash D \subseteq V(H)$. Thus, $u \in V(H)$. In the join graph $G+H$, every vertex of H is adjacent to every vertex of G. Since $C \subseteq V(G)$ and $C \neq \emptyset$, it follows that u is adjacent to some vertex in $C \subseteq D$. Hence, every vertex in $V(G+H)\backslash D$ is dominated by D. Therefore, D is a dominating set of $G+H$. Next, we show that C and S_G are connected. By assumption, C is a connected dominating set of $G+H$, hence $\langle V(G)\backslash C\rangle$ is connected. Also, since S_G is an inverse connected dominating set of G, it is a connected dominating set of G, and thus $\langle V(G)\backslash S_G\rangle$ is connected. Since G is a subgraph of $G+H$, it follows that S_G remains connected in $G+H$. Thus, both C and S_G induce connected subgraphs in $G+H$. Since $S_G \subseteq V(G)\backslash C$, we have $C \cap S_G = \emptyset$. Hence, C and S_G are disjoint connected dominating sets of $G+H$, and therefore $D = C \cup S_G$ is a disjoint connected dominating set of $G+H$. ☐

Lemma 2.15: Let G and H be nontrivial connected graphs. A subset $D \subseteq V(G+H)$ is a disjoint connected dominating set of $G+H$ if $D = C \cup S_H$, where C is a minimum connected dominating set of both H and $G+H$, and S_H is an inverse connected dominating set of H with respect to C.

***Proof*:** Let G and H be nontrivial connected graphs. Suppose that $D = C \cup S_H$, where $C \subseteq V(H)$ is a minimum connected dominating set of both H and $G+H$, and $S_H \subseteq V(H)\backslash C$ is an inverse connected dominating set of H with respect to C. We show that D is a disjoint connected dominating set of $G+H$. First, we prove that D is a dominating set of $G+H$. Let $u \in V(G+H)\backslash D$. Then

$$V(G+H)\backslash D = (V(G)\cup V(H))\backslash(C\cup S_H) = V(G)\cup(V(H)\backslash(C\cup S_H))$$

***Case 1*:** $u \in V(G)$. Since $G+H$ is the join of G and H, every vertex of G is adjacent to every vertex H. In particular, u is adjacent to every vertex of $C \subseteq V(H)$. Hence, u is dominated by $C \subseteq D$.

***Case 2*:** $u \in V(H)\backslash(C\cup S_H)$. Since S_H is an inverse connecetd dominating set of H, it follows that S_H is a connected dominating set of H. Thus, there exists a vertex $w \in S_H$ such that $uw \in E(H)$. Hence, u is dominated by $S_H \subseteq D$.

Thus, every vertex in $V(G+H)\backslash D$ is adjacent to a vertex in D, and therefore D is a dominating set of $G+H$. Next, we show that D induces a connected subgraph. Since C is a connected dominating set of H, the induced subgraph $\langle V(H)\backslash C\rangle$, and hence $\langle V(G+H)\rangle\backslash C$, is connected. Similarly, since S_H is a connected dominating set of H, the induced subgraph $\langle V(H)\backslash S_H\rangle$, and thus $\langle V(G+H)\backslash S_H\rangle$, is connected. Moreover, since H is connected and both C and S_H are nonempty subsets of $V(H)$, there exists a path in H joining some vertex of C to some vertex of S_H. Consequently, the induced subgraph $\langle V(G+H)\backslash D\rangle$ is connected. Finally, since $S_H \subseteq V(H)\backslash C$, we have $C\cap S_H = \emptyset$.Therefore, C and S_H are disjoint subsets whose union $D = C\cup S_H$ forms a connected dominating set of $G+H$. Hence, D is a disjoint connected dominating set of $G+H$. ☐

Lemma 2.16: Let G and H be nontrivial connected graphs. A subset $D \subseteq V(G+H)$ is a disjoint connected dominating set of $G+H$ if $D = C\cup S$, where C is a minimum connected dominating set of $G+H$, and S is an inverse connected dominating set of $G+H$ with respect to C.

***Proof*:** Let G and H be nontrivial connected graphs. Suppose that $D = C\cup S$, where C is a minimum connected dominating set of $G+H$, and $S \subseteq V(G+H)\backslash C$ is an inverse connected dominating set of $G+H$ with respect to C. We first show that D is a dominating set of $G+H$. Let $u \in V(G+H)\backslash D$. Then $V(G+H)\backslash D = V(G+H)\backslash(C\cup S)$. Since S is a dominating set of $G+H$, there exists a vertex $v \in S$ such that $uv \in E(G+H)$. Hence, every vertex in $V(G+H)\backslash D$ is adjacent to a vertex in $S \subseteq D$. Thus, D is a dominating set of $G+H$. Next, we show that D induces a connected subgraph. By assumption, C is a connected dominating set of $G+H$, so $\langle V(G+H)\backslash C\rangle$ is connected. Also, since S is an inverse connected dominating set of $G+H$, it is a connected dominating set of $G+H$, and hence $\langle V(G+H)\backslash S\rangle$ is connected. Since both C and S are connected dominating set of $G+H$ there must exist at least one edge between a vertex in C and a vertex in S; otherwise, $G+H$ would be disconnected, which is contradiction. Therefore, $\langle V(G+H)\backslash D\rangle$ is connected. Since $S \subseteq V(G+H)\backslash C$, we have $C\cap S = \emptyset$.Therefore, C and S are disjoint connected dominating sets of $G+H$, and hence $D = C\cup S$ is a disjoint connected dominating set of $G+H$. ☐

Theorem 2.17: Let G and H be nontrivial connected graphs. A subset $D \subseteq V(G+H)$ is a disjoint connected dominating set of $G+H$ if and only if one of the following holds.

i. $D = C\cup S_G$, where C is a minimum connected dominating set of G and of $G+H$, and S_G is an inverse connected dominating set of G with respect to C;
ii. $D = C\cup S_H$, where C is a minimum connected dominating set of H and of $G+H$, and S_H is an inverse connected dominating set of H with respect to C;
iii. $D = C\cup S$, where C is a minimum connected dominating set of $G+H$, and S is an inverse connected dominating set of $G+H$ with respect to C.

***Proof*:**
$\Longrightarrow$Suppose that D is a disjoint connected dominating set of $G+H$. Then there exist two disjoint subsets C and S such that $D = C\cup S,\ C\cap S = \emptyset$,and both $C$and S are connected dominating sets of $G+H$. Let C be a minimum connected dominating set of $G+H$ contained in D. Then $S = D\backslash C \subseteq V(G+H)\backslash C$ is an inverse connected dominating set of $G+H$ with respect to C. Now consider the structure of S with respect to the partition $V(G+H) = V(G)\cup V(H)$.

***Case 1*:** $S \subseteq V(G)$. Then $S = S_G$, where S_G is an inverse connected dominating set of G with respect to C. Since C dominates $G+H$, in particular it dominates G, and by minimality it is also a minimum connected dominating set of G. Hence, condition (1) holds.

***Case 2*:** $S \subseteq V(H)$. Similarly, $S = S_H$, where S_H is an inverse connected dominating set of H with respect to C. Thus, condition (2) holds.

***Case 3*:** S contains vertices from both $V(G)$ and $V(H)$. Then S is naturally an inverse connected dominating set of $G+H$ with respect to C, and condition (3) holds.
Thus, at least one of the conditions (1), (2), or (3) must hold.
⟸Conversely, suppose that one of the conditions (1), (2), or (3) holds. If (1) holds, then by Lemma 2.14, $D=C\cup S_G$ is a disjoint connected dominating set of $G+H$. If (2) holds, then by Lemma 2.15, $D=C\cup S_H$ is a disjoint connected dominating set of $G+H$. If (3) holds, then by Lemma 2.16, $D=C\cup S$ is a disjoint connected dominating set of $G+H$. In all cases, D is a disjoint connected dominating set of $G+H$. Therefore, D is a disjoint connected dominating set of $G+H$ if and only if one of the conditions (1), (2), or (3) holds. ☐

Corollary 2.18: Let G and H be nontrivial connected graphs. Then

$$\gamma\gamma_c(G+H)=\gamma_c(G+H)+\gamma_{c^{-1}}(G+H)$$

Proof: Let G and H be nontrivial connected graphs, and consider the join graph $G+H$. Let D be a minimum disjoint connected dominating set of $G+H$. Then there exist disjoint subsets $C,S\subseteq V(G+H)$ such that $D=C\cup S,\ C\cap S=\emptyset$,where both C and S are connected dominating sets of $G+H$, and $|D|=\gamma\gamma_c(G+H)$. Choose C to be a minimum connected dominating set of $G+H$. Then $|C|=\gamma_c(G+H)$. Since $S=D\backslash C\subseteq V(G+H)\backslash C$ and S is a connected dominating set, it follows that S is an inverse connected dominating set with respect to C. By minimality of D, we may take S to be of minimum cardinality, hence $|S|=\gamma_{c^{-1}}(G+H)$. Therefore,

$$\gamma\gamma_c(G+H)=|D|=|C|+|S|=\gamma_c(G+H)+\gamma_{c^{-1}}(G+H)$$

Conversely, let C be a minimum connected dominating set of $G+H$ and let $S\subseteq V(G+H)\backslash C$ be a minimum inverse connected dominating set with respect to C. Then $C\cap S=\emptyset$,and both C and S are connected dominating sets of $G+H$. Hence, $D=C\cup S$ is a disjoint connected dominating set of $G+H$, and $|D|=|C|+|S|=\gamma_c(G+H)+\gamma_{c^{-1}}(G+H)$. By minimality, it follows that
$\gamma\gamma_c(G+H)=\gamma_c(G+H)+\gamma_{c^{-1}}(G+H)$. ☐

III. Conclusion

This paper introduced the concept of disjoint connected domination in graphs and determined the disjoint connected domination number $\gamma\gamma_c(G)$ for several classes of graphs. The results show that the existence and value of $\gamma\gamma_c(G)$ depend on the structure of the graph, with some graphs admitting such sets while others do not. We also characterized disjoint connected dominating sets in the join of two nontrivial connected graphs and established a relationship with connected domination parameters. These findings contribute to domination theory and may have applications in network design and fault-tolerant systems.

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