Weakly Convex Restrained Domination on Graphs

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Abstract: Let G be a connected simple graph. A dominating set $\mathrm{S} \subseteq \mathrm{V(G)}$ is a restrained dominating set if every vertex not in S is adjacent to a vertex in S and to a vertex in $\mathrm{V(G)} \setminus \mathrm{S}$. Alternatively, a dominating set $\mathrm{S} \subseteq \mathrm{V(G)}$ is a restrained dominating set if $\mathrm{N[S]} = \mathrm{V(G)}$ and $\langle \mathrm{V(G)} \setminus \mathrm{S} \rangle$ has no isolated vertices. A restrained dominating set S is called a weakly convex restrained dominating set if for every two vertices $\mathrm{u}, \mathrm{v} \in \mathrm{S}$, there exists a $\mathrm{u} - \mathrm{v}$ geodesic whose vertices belong to S. The minimum cardinality of a weakly convex restrained dominating set of G, denoted by $\gamma_{\mathrm{wcr}}(\mathrm{G})$, is called the weakly convex restrained domination number of G. This paper initiates the study of weakly convex restrained domination in graphs and determines the weakly convex restrained domination number of some special graphs. Furthermore, it presents a characterization of the weakly convex restrained dominating set in the join and corona of two nontrivial connected graphs.

Keywords: dominating set, restrained dominating set, weakly convex dominating set, weakly convex restrained dominating set

I. Introduction

Graph theory is a fundamental area of discrete mathematics with wide-ranging applications in communication networks, transportation systems, biological modeling, social networks, and optimization problems. One of the most extensively studied topics in graph theory is domination, which models the concept of influence, control, or coverage within a network. A comprehensive treatment of domination theory and its variants is provided in the classical monograph of Haynes, Hedetniemi, and Slater [1].

Let $G = (V(G),\ E(G))$ be a simple graph. A subset $S \subseteq V(G)$ is called a dominating set if every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S. Since its introduction, domination has evolved into a rich field with numerous variations motivated by both theoretical and applied considerations [2, 3]. These variations impose additional constraints on dominating sets to capture more refined structural properties of graphs.

One important variant is restrained domination, introduced to ensure that vertices outside the dominating set retain a certain level of connectivity. A dominating set $S \subseteq V(G)$ is said to be a restrained dominating set if every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S and at least one vertex in $V(G) \setminus S$. This concept guarantees that the subgraph induced by $V(G) \setminus S$ has no isolated vertices, making it particularly useful in applications where redundancy or backup connections are required [4, 5]. Restrained domination in graphs has been further investigated in several studies, including works on convex and secure variants, inverse domination, and domination in graph operations. Related results can be found in the papers [6, 7, 8, 9, 10, 11, 12, 13].

Another direction in graph theory involves the study of convexity based on distances. Convexity in graphs is closely related to geodesics (shortest paths). A subset S of G is called weakly convex if, for every pair of vertices $u, v \in S$, there exists a $u - v$ geodesic whose vertices lie entirely in S. This notion, studied by several authors, plays an important role in routing, navigation, and network optimization problems [14, 15]. The integration of domination and convexity concepts has led to the development of new domination parameters that incorporate both coverage and structural cohesion. In particular, weakly convex domination and its variants have attracted increasing attention in recent years due to their ability to model efficient communication and connectivity constraints simultaneously [16, 17].

Motivated by these developments, this paper focuses on the concept of weakly convex restrained domination. A restrained dominating set $S \subseteq V(G)$ is called a weakly convex restrained dominating set if it satisfies the weak convexity condition. The minimum cardinality of such a set is called the weakly convex restrained domination number of G, denoted by $\gamma_{wcr}(G)$. This parameter combines the coverage requirement of domination, the stability condition of restrained domination, and the structural constraint of geodesic convexity.

The study of domination parameters on specific graph classes has proven to be an effective approach in understanding their structural behavior. Various results have been established for paths, cycles, complete graphs, bipartite graphs, and other special graph families [18, 19, 20]. However, the interaction between restrained domination and weak convexity remains relatively unexplored. In particular, while restrained domination and weakly convex domination have been studied independently, their combined behavior under structural constraints and graph operations such as join and corona remains largely unexplored.

In this paper, the weakly convex restrained domination number is investigated for several classes of graphs, including cycles, complete graphs, fan graphs, wheel graphs, and complete bipartite graphs. Furthermore, a characterization of weakly convex restrained dominating sets in the join and corona of two graphs is provided. Exact values are determined, and structural properties of weakly convex restrained dominating sets are analyzed. The results obtained contribute to the growing literature on domination theory and provide further insights into the interplay between domination, convexity, and graph structure.

II. Results

Definition 2.1: [1] Let $G = \big(V(G). E(G)\big)$ be a simple graph. A subset $S \subseteq V(G)$ is called a *dominating set* of G if every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S. That is, for every vertex $v \in V(G) \setminus S$, there exists a vertex $u \in S$ such that $uv \in E(G)$.

Definition 2.2: [4] A dominating set $S \subseteq V(G)$ is called a *restrained dominating set* of G if every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S and at least one vertex in $V(G) \setminus S$. Alternatively, a dominating set $S \subseteq V(G)$ is a restrained dominating set if $N[S] = V(G)$ and $\langle V(G) \setminus S \rangle$ is a subgraph without isolated vertices.

Definition 2.3: A restrained dominating set S of $V(G)$ is called a *weakly convex restrained dominating set* of G if for every two vertices $u, v \in S$, there exists a $u - v$ geodesic whose vertices belong to S. The minimum cardinality of a weakly convex restrained dominating set of G, denoted by $\gamma_{wcr}(G)$, is called the weakly convex restrained domination number of G.

Definition 2.4: A *cycle* $C_n = a_1 a_2 \cdots a_n a_1$ is a graph with $V(C_n) = \{a_1, a_2, \dots, a\_n\}$ and $E(C_n) = \{a_1 a_2, a_2 a_3, \dots, a_{n-1} a_n, a_n a_1\}$.

The weakly convex restrained domination number of cycles is first determined.

Proposition 2.5: *Let $G = C_n$ for all $n \geq 3$. Then,*

$$\gamma_{wcr}(G) = \begin{cases} n-2, \ if\ 3 \leq n \leq 6 \\ none, \ if\ n \geq 7. \end{cases}$$

Proof: Let $C_n = a_1 a_2 \cdots a_n a_1$, where $n \geq 3$ and indices are taken modulo n.

Case 1: 3 ≤ n ≤ 6.

Let $S = V(C_n) \setminus \{a_i, a_{i+1}\}$ for some $i \in \{a_i, a_{i+1}\}$.The vertices not in S are a_1 and a_{i+1}. Since a_1 is adjacent to $a_{i+1} \in S$ and a_{i+1} is adjacent to $a_{i+2} \in S$, every vertex in $V(C_n) \setminus S$ is adjacent to a vertex in S. Hence, S is a dominating set. The induced subgraph $\langle V(C_n) \setminus S \rangle = \langle \{a_i a_{i+1}\} \rangle$. Thus, it has no isolated vertex, and S is a restrained dominating set.The subgraph induced by S is a path on $n-2$ vertices. For $3 \leq n \leq 6$, for any two vertices $u, v \in S$, there exists a shortest $u - v$ path in C_n that avoids both a_i and a_{i+1}. Hence, there exists a $u - v$ geodesic whose vertices are all in S. Thus, S is a weakly convex restrained dominating set. Therefore, $\gamma_{wcr}(C_n) \leq n-2$. To establish equality, let $S' \subseteq V(C_n)$ be a weakly convex restrained dominating set. Since S' is restrained, the induced subgraph $\langle V(C_n) \setminus S' \rangle$ has no isolated vertex. Hence, $|V(C_n) \setminus S'| \neq 1$. If $|V(C_n) \setminus S'| \geq 3$, then there exist three consecutive vertices not in S', say a_j, a_{j+1}, and a_{j+2}. Consider $a_{j-1}, a_{j+3} \in S'$. Every shortest path between these vertices contains at least one of a_j, a_{j+1}, a_{j+2}, and hence is not contained entirely in S', contradicting weak convexity. Therefore, $|V(C_n) \setminus S'| = 2$, and so $|S'| = n-2$. Hence, $\gamma_{wcr}(C_n) = n-2$ for $3 \leq n \leq 6$.

Case 2: n ≥ 7.

Assume that $S \subseteq V(C_n)$ is a weakly convex restrained dominating set. Since S is restrained, the induced subgraph $\langle V(C_n) \setminus S \rangle$ has no isolated vertex. Thus, there exist at least two consecutive vertices not in S, say a_i and a_{i+1}. Since $n \geq 7$, the vertices a_{i-2} and a_{i+3} belong to S. Every shortest path between a_{i-2} and a_{i+3} contains at least one of a_i or a_{i+1}. Hence, no $a_{i-2} - a_{i+3}$ geodesic is contained entirely in S, contradicting the weak convexity of S. Therefore, no weakly convex restrained dominating set exists in C_n for $n \geq 7$.

Consequently, $\gamma_{wcr}(C_n)$ is none for $n \geq 7$.

■

Definition 2.6: A *complete graph* K_n is a graph of order n where every pair of its vertices is adjacent.

Definition 2.7: A *path* $P_n = a_1 a_2 \dots a_n$ is a graph with vertex set $V(P_n) = \{a_1, a_2, \dots, a_n\}$ and edge set $E(P_n) = \{a_1 a_2, a_2 a_3, \dots, a_{n-1} a_n\}$.

Definition 2.8: The *fan* F_n is the graph with $V(F_n) = \{a_0, a_1, \dots, a_n\}$ and $E(F_n) = \{a_1 a_2, a_2 a_3, \dots, a_{n-1} a_n\} \cup \{a_0 a_i : i = 1, 2, \dots, n\}$.

The following result establishes the weakly convex restrained domination number of complete graphs and fan graphs.

Proposition 2.9: *Let* $G = K_n$ *or* $G = F_n$*, where* $F_n = K_1 + P_n$*, for all* $n \geq 2$*. Then,* $\gamma_{wcr}(G) = 1$.

Proof: Let G be either K_n or $F_n = K_1 + P_n$, where $n \geq 2$.

Case 1: $\mathbf{G = K_n}$, where $\mathbf{n \geq 2}$.

Let $S = \{v\}$, where $v \in V(K_n)$. Since K_n is a complete graph, every vertex in $V(K_n) \setminus S$ is adjacent to v. Hence, S is a dominating set of G.Moreover, the induced subgraph $\langle V(K_n) \setminus S \rangle$ is isomorphic to K_{n-1}, which has no isolated vertices. Thus, S is a restrained dominating set. Since $|S| = 1$, there are no distinct vertices $u, v \in S$. Therefore, the condition that every pair of vertices in S is connected by a geodesic contained in S is vacuously satisfied. Hence, S is a weakly convex restrained dominating set of K_n. Thus, $\gamma_{wcr}(K_n) = 1$.

Case 2: $\mathbf{G = F_n = K_1 + P_n}$, where $\mathbf{n \geq 2}$.

Let $V(F_n) = \{a_0, a_1, a_2, \dots, a_n\}$, where $a_1 a_2 \dots a_n$ forms a path and a_0 is adjacent to every $a_i, 1 \leq i \leq n$. Let $S = \{a_0\}$. Since a_0 is adjacent to every vertex of P_n, every vertex in $V(F_n) \setminus S$ is adjacent to a vertex in S. Hence, S is a dominating set.The induced subgraph $\langle V(F_n) \setminus S \rangle = \langle \{a_1, a_2, \dots, a_n\} \rangle$ is the path P_n, which has no isolated vertices for $n \geq 2$. Thus, S is a restrained dominating set. Since $|S| = 1$, the weak convexity condition is again satisfied vacuously. Therefore, S is a weakly convex restrained dominating set of F_n. Thus, $\gamma_{wcr}(F_n) = 1$.

Since in both cases a weakly convex restrained dominating set of cardinality one exists, and no dominating set can have cardinality less than one, it follows that $\gamma_{wcr}(G) = 1$. ■

Definition 2.10: The *wheel* W_n is the graph with $V(W_n) = \{a_1, a_2, \dots, a_n\}$ and
$E(W_n) = \{a_1 a_2, a_2 a_3, \dots, a_{n-1} a_n, a_n a_1\} \cup \{a_0 a_i : i = 1, 2, \dots, n\}$.

The weakly convex restrained domination number of wheel graphs is then determined.

Proposition 2.11: *Let* $G = W_n$*, where* $W_n = K_1 + C_n$*, for all* $n \geq 3$*. Then,* $\gamma_{wcr}(G) = 1$.

Proof: Let $G = W_n = K_1 + C_n$, where $n \geq 3$. Let $V(W_n) = \{a_0, a_1, a_2, \dots, a_n\}$, where $a_1 a_2 \dots a_n a_1$ forms a cycle and a_0 is adjacent to every vertex $a_i, 1 \leq i \leq n$.Let $S = \{a_0\}$. Since a_0 is adjacent to every vertex of the cycle C_n, every vertex in $V(G) \setminus S$ is adjacent to a vertex in S. Hence, S is a dominating set of G.The induced subgraph $\langle V(G) \setminus S \rangle = \langle \{a_1, a_2, \dots, a_n\} \rangle$ is isomorphic to C_n, which contains no isolated vertices for $n \geq 3$. Therefore, S is a restrained dominating set. Since $|S| = 1$, there are no distinct vertices $u, v \in S$, and the condition that every pair of vertices in S is connected by a geodesic contained in S holds vacuously. Thus, S is a weakly convex restrained dominating set of G. Therefore, $\gamma_{wcr}(W_n) = 1$.

Definition 2.12: A *complete bipartite graph* is a graph whose vertex set can be partitioned into V_1 and V_2 such that every edge joins a vertex in V_1 with a vertex in V_2 and every vertex in V_1 is adjacent with every vertex in V_2.

The weakly convex restrained domination number of complete bipartite graphs is next established.

Proposition 2.13: *Let* $G = K_{m,n}$*, where* $K_{m,n} = \overline{K}_m + \overline{K}_n$ *for all* $m, n \geq 2$*. Then,* $\gamma_{wcr}(G) = 2$.

Proof: Let $G = K_{m,n}$, where $m, n \geq 2$, and let $V(G)$ be partitioned into two independent sets V_1 and V_2 with $|V_1| = m$ and $V_2 = n$. Thus, every vertex in V_1 is adjacent to every vertex in V_2 and there are no edges between vertices within the same part.Now, let $x \in V_1$ and $y \in V_2$. Consider the set $S = \{x, y\}$. Every vertex in $V_1 \setminus \{x\}$ is adjacent to $y \in S$, and every vertex in $V_2 \setminus \{y\}$ is adjacent to $x \in S$. Hence, every vertex in $V(G) \setminus S$ is adjacent to a vertex in S, and so S is a dominating set. Consider the induced subgraph $\langle V(G) \setminus S \rangle$, whose vertex set is $V_1 \setminus \{x\} \cup V_2 \setminus \{y\}$. Since $m, n \geq 2$, both $V_1 \setminus \{x\}$ and $V_2 \setminus \{y\}$ are nonempty, and every vertex in one part remains adjacent to all vertices in the other part. Thus, no vertex in $\langle V(G) \setminus S \rangle$ is isolated, and S is a restrained dominating set. Now, since x and y are adjacent, the unique $x - y$ geodesic is the path xy whose vertices lie entirely in S. Hence, S is weakly convexrestrained dominating set. Therefore, $\gamma_{wcr}(G) \leq 2$.

Suppose, to the contrary, that there exists a weakly convex restrained dominating set S with $|S| = 1$, say $S = \{v\}$. If $v \in V_1$, then v is not adjacent to any vertex in $V_1 \setminus \{v\}$, and hence S does not dominate G. Similarly, if $v \in V_2$, then vertices in $V_2 \setminus \{v\}$ are not dominated. This contradicts the assumption that S is a dominating set. Thus, no singleton set is a dominating set, and so $\gamma_{wcr}(G) \geq 2$.

Combining both in equalities yield s$\gamma_{wcr}(G) = 2$. ■

Definition 2.14 [21] Let G and H be graphs. The *join* of G and H, denoted by $G + H$, is the graph obtained from the union of G and H by adding edges joining every vertex of G to every vertex of H. Formally, $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup \{uv \mid u \in V(G),\ v \in V(H)\}$.

The following result characterized the weakly convex restrained dominating sets in the join of two graphs.

Theorem 2.15: *Let G and H be nontrivial connected graphs. Then $S \subseteq V(G + H)$ is a weakly convex restrained dominating set of $G + H$ if and only if one of the following conditions is satisfied:*

- *(i)* *S is a connected dominating set of G (or H) and $d_G(x, y) \leq 2$ for all $x, y \in S$;*
- *(ii)* *$S = V(H) \cup S_G$, where $S_G \subset V(G)$;*
- *(iii)* *$S = V(G) \cup S_H$, where $S_H \subset V(H)$;*
- *(iv)* *$S = S_G \cup S_H$, where $\langle S_G \rangle$ and $\langle S_H \rangle$ are complete subgraphs of G and H, respectively;*
- *(v)* *$S = S_G$ is a dominating set of G and $\langle S_G \rangle$ is complete; or*
- *(vi)* *$S = S_H$ is a dominating set of H and $\langle S_H \rangle$ is complete.*

Proof: Assume that $S \subseteq V(G + H)$ is a weakly convex restrained dominating set of $G + H$. Since $G + H$ is the join of G and H, every vertex of G is adjacent to every vertex of H. Hence, for any $u \in V(G)$ and $v \in V(H)$, $d_{G+H}(u, v) = 1$, and for vertices in the same part, the distance is at most 2.

Case 1: $\mathbf{S \subseteq V(G)}$ (the case $\mathbf{S \subseteq V(H)}$ is analogous).

Since S is a restrained dominating set of $G + H$, it follows that S is a dominating set of G. The vertices of H are dominated automatically by S due to the join. Now, for every $x, y \in S$, there exists an $x - y$ geodesic contained in S. Since distances are at most 2, this implies that either x and y are adjacent or share a common neighbor in S. Hence, $\langle S \rangle$ is connected and $d_{G+H}(x, y) \leq 2$ for all $x, y \in S$. This establishes (i). In particular, if $\langle S \rangle$ is complete, then (v) or (vi) holds.

Case 2: $\mathbf{S \cap V(G) \neq \emptyset}$ and $\mathbf{S \cap V(H) \neq \emptyset}$.

Write $S = S_G \cup S_H$, where $S_G \subseteq V(G)$ and $S_H \subseteq V(H)$. If $V(H) \subseteq S$, then $S = V(H) \cup S_G$, which satisfies (ii). Similarly, if $V(G) \subseteq S$, then (iii) holds.Assume that neither $V(G) \subseteq S$ nor $V(H) \subseteq S$. Consider any $x, y \in S_G$. Since $d_{G+H}(x, y) \leq 2$, any geodesic between them must lie entirely in S. If x and y are not adjacent in G, then a shortest path between them would pass through a vertex in $V(H) \setminus S_H$, contradicting weak convexity. Hence, x and y are adjacent, and $\langle S_G \rangle$ is complete. Similarly, $\langle S_H \rangle$ is complete.Thus, (iv) holds.

Conversely, suppose that S satisfies any one of the conditions (i)-(vi).

Case 1: Suppose that S satisfies condition (i). Then S is a connected dominating set of G (or H) and $d_G(x, y) \leq 2$ for all $x, y \in S$. Since every vertex of H is adjacent to every vertex of G in $G + H$, itfollows that S dominates $G + H$. Let $u \in V(G + H) \setminus S$. Let $u \in V(G)$. Since S is a dominatingset of G, there exists $w \in S$ such that $uw \in E(G)$. Moreover, because G is nontrivial and connected, u also has a neighbor in $V(G) \setminus S$. If $u \in V(H)$,

then u is adjacent to every vertex of S and alsoto vertices in$V(H) \setminus S$. Hence, every vertex outside$S$ hasneighbors both in S and outside S, so S is a restrained dominating set of $G + H$.
Now let $x, y \in S$. Since $d_G(x, y) \leq 2$, there existsan $x - y$ geodesic in G of length at most 2. Because$S$ is connected, such a geodesic can be chosen entirely in S. Hence, S is weakly convex. Therefore, S is a weakly convex restrained dominating set of $G + H$.

Case 2: Suppose that S satisfies condition (ii). Then $S = V(H) \cup S_G$, where $S_G \subseteq V(G)$. Since $V(H) \subseteq S$, every vertex of $V(G) \setminus S_G$ is adjacent to allvertices of $V(H)$, and hence is dominated by S. Thus, S is a dominating set of $G + H$. Let$u \in V(G + H) \setminus S$. Then$u \in V(G) \setminus S_G$.Since u is adjacent to every vertex of$V(H) \subseteq S$ andalsohasneighborsin$V(G) \setminus S_G$, the restrained condition holds. Let $x, y \in S$. If both vertices belong to $V(H)$, then either they are adjacent in H or there exists a path$x - w - y$with$w \in V(H) \subseteq S$. If one vertex belongs to $V(H)$ and the other belongs to S_G, then theyare adjacent in $G + H$. If both vertices belong to S_G, then they have distance at most 2 through a vertexof$V(H) \subseteq S$. Hence, every geodesic between vertices of S is contained in S, proving that $S$is weakly convex. Therefore, S is a weaklyconvex restrained dominating set of $G + H$.

Case 3: Suppose that S satisfies condition (iii). Then $S = V(G) \cup S_H$, where $S_H \subseteq V(H)$. By similar arguments used in Case 2, S is a weakly convex restraineddominating set of $G + H$.

Case 4: Suppose that S satisfies condition (iv). Then $S = S_G \cup S_H$, where $\langle S_G \rangle$ and $\langle S_H \rangle$ are complete subgraphs of G and H, respectively. Every vertex in $V(G) \setminus S_G$ is adjacent to every vertex of S_H, while every vertex in $V(H) \setminus S_H$ is adjacent to every vertex of S_G. Hence, S dominates $G + H$. Let $u \in V(G + H) \setminus S$. If $u \in V(G) \setminus S_G$, then u is adjacent to vertices of S_H and alsoto vertices in $V(G) \setminus S_G$. Similarly, if $u \in V(H) \setminus S_H$, then u is adjacent to vertices of S_G and also to vertices in $V(H) \setminus S_H$. Thus, the restrained condition holds.Now let$x, y \in S$. If both belong toS_G, then they are adjacent since $\langle S_G \rangle$ is complete.Similarly, vertices in S_H are pairwiseadjacent. If one vertex belongs toS_G and the other belongs to S_H, then they are adjacent by the definition of the join. Hence, every geodesic between vertices of Slies entirely in S, so S is weakly convex. Therefore, S is a weakly convex restrained dominating set of $G + H$.

Case 5: Suppose that S satisfies condition (v). Then $S = S_G$, where S_G is a dominating set of G and $\langle S_G \rangle$ is complete. Since every vertex of H is adjacentto every vertex of G, it follows that S dominates $G + H$. Let$u \in V(G + H) \setminus S$. If$u \in V(G) \setminus S_G$, then u has a neighbor inS_G since S_G is a dominating set of G, and it also has neighbors outside S. If $u \in V(H)$, then u is adjacent to every vertex of S_G and also to vertices of H. Hence, there strained condition holds.

For any $x, y \in S_G$, the completeness of S_G implies that $xy \in E(G)$, so every geodesic between vertices of S lies entirely in S. Therefore, S is weakly convex, and consequently S is a weaklyconvex restrained dominating set of $G + H$.

Case 6: Suppose that S satisfies condition (vi). Then $S = S_H$, where S_H is a dominating set of H and $\langle S_H \rangle$ is complete. By similar arguments used in Case 5, S is a weakly convex restrained dominating set of $G + H$. ■

The following result is an immediate consequence of Theorem 2.15.

Corollary 2.16: *Let G and H be nontrivial connected graphs. Then,*

$$\gamma_{wcr}(G + H) = \begin{cases} 1, \ if\ \gamma(G) = 1\ or\ \gamma(H) = 1, \\ 2,\ otherwise. \end{cases}$$

Proof: Suppose that $\gamma(G) = 1$. Then there exists a vertex$v \in V(G)$ such that v is adjacent to every vertex of G. Since $G + H$ is the join of G and H, the vertex v is also adjacent to every vertex of H. Hence, $S = \{x\}$is a dominating set of $G + H$. Moreover, since G and H are nontrivial and connected, the induced subgraph $\langle V(G + \mathcal{H} \setminus \mathcal{S}$ has no isolated vertices. Trivially, $\mathcal{S}$ is weakly convex. Thus, $\mathcal{S}$ is a weakly convex restrained dominating set of $G + H$, and therefore γ_{wcr} (G+H) $= 1$. A similar argument holds if $\gamma(H) = 1$.

Now suppose that $\gamma(G) > 1$and $\gamma(H) > 1$. Then no single vertex of G (respectively, H) dominates G (respectively, H). Hence, no single vertex dominates $G + H$, and so $\gamma_{wcr}(G + H) \geq 2$.

Let $x \in V(G)$ and $y \in V(H)$. Consider $S = \{x, y\}$. Since every vertex of $G$is adjacent to y and every vertex of H is adjacent to x, it follows that S is a dominating set of $G + H$. Let $v \in V(G + H) \setminus S$. Then v has a neighbor in S and also a neighbor in $V(G + H) \setminus S$, since both G and H are connected and nontrivial. Hence, S satisfies the restrained condition. Finally, since x and y are adjacent in $G + H$, every $x - y$ geodesic is contained in S. Thus, S is a weakly convex restrained dominating set of $G + H$. Therefore, $\gamma_{wcr}(G + H) \leq 2$.

Combining these results, it follows that $\gamma_{wcr}(G + H) = 2$. ■

Definition 2.17: [22] Let G and H be graphs and suppose that G has n vertices. The *corona* of G and H, denoted by $G \circ H$, is the graph obtained by taking one copy of G together with n copies of H, and then joining the ith vertex of G to every vertex in the ith copy of H, for each$i = 1, 2, \ldots, n$.

The following result characterized the weakly convex restrained dominating sets in the corona of two graphs.

Theorem 2.18: *Let G and H be connected nontrivial graphs. The subset $S \subset V(G \circ H)$ is a weakly convex restrained dominating set of $G \circ H$ if and only if one of the following conditions is satisfied:*

- *(i)* $S = V(G)$;
- *(ii)* $S = V(G) \cup \left(\bigcup_{v \in S_G} V(H^v)\right)$*for some*$S_G \subset V(G)$;
- *(iii)* $S = V(G) \cup \left(\bigcup_{v \in S_G} S_v\right)$ *for some*$S_G \subseteq V(G)$ *and* $S_v \subseteq V(H^v)$*; or*
- *(iv)* $S = V(G) \cup \left(\bigcup_{v \in S_G} V(H^v)\right) \cup \left(\bigcup_{x \in V(G) \setminus S_G} S_x\right)$*for some*$S_G \subset V(G)$ *and*$S_x \subset V(H^x)$, $x \in V(G) \setminus S_G$.

Proof: Suppose that $S \subseteq V(G \circ H)$ is a weakly convex restrained dominating set of $G \circ H$.

Suppose, to the contrary, that there exists$v \notin V(G)$ such that$v \notin S$. Since S is a dominating set, there exists a vertex $u \in S$ such that $uv \in E(G \circ H)$. By the definition of the corona, this implies that $u \in V(H^v)$. Now consider any vertex $x \in V(H^v) \setminus S$. The neighbors of $x$lie only in $H^v$and the vertex v. Since $v \notin S$, it follows that x has no neighbor in S, contradicting the fact that S is a dominating set. Hence, $V(G) \subseteq S$. Thus, S can be written in the form $S = V(G) \cup \left(\bigcup_{v \in V(G)} T_v\right)$, where $T_v \subseteq V(H^v)$ for each $v \in V(G)$.

Depending on the choice of the subsets T_v, the set S assumes one of several forms. If$T_v = \emptyset$ for all$v \in V(G)$, then no vertices from the copies $H^v$are included, and consequently $S = V(G)$, which corresponds to case (i). If $T_v = V(H^v)$ for all $v \in S_G \subset V(G)$ and $T_v = \emptyset$ otherwise, then all vertices in the copies H^v corresponding to vertices in S_G are included while the remaining copies contribute no vertices, yielding the structure described in case (ii). If $T_v = S_v \subseteq V(H^v)$ for all $v \in S_G \subseteq V(G)$ and $T_v = \emptyset$ otherwise, then only partial subsets of the copies H^vcorresponding to vertices in $S_G$are included, leading to the form given in case (iii). In the most general situation, if $T_v = V(H^v)$ for $v \in S_G$ and $T_v = S_v \subseteq V(H^v)$ for $v \in V(G) \setminus S_G$, then S contains full copies for vertices in S_G and partial selections from the remaining copies, resulting in the structure described in case (iv). Hence, S satisfies one of the conditions (i)-(iv).

Conversely, suppose that$S$ satisfies any of the conditions (i)-(iv).

Case 1: Suppose that S satisfies condition (i). Then $S = V(G)$. Every vertex in each copy H_v is adjacent to the corresponding vertex $v \in V(G) = S$. Hence, S dominates $G \circ H$. Let $x \in V(G \circ H) \setminus S$. Then $x \in V(H_v)$ for some $v \in V(G)$. Since x is adjacent to $v \in S$ and also to vertices in $H_v \setminus S$, the restrained condition holds. Now let $u, w \in S = V(G)$. Any shortest $u - w$path in $G \circ H$lies entirely in $V(G)$, since vertices from the copies H_v are pendant relative to the base graph. Thus, every geodesic between vertices of S is contained in S. Therefore, S is weakly convex, and hence S is a weakly convex restrained dominating set of $G \circ H$.

Case 2: Suppose that S satisfies condition (ii). Then $S = V(G) \cup \left(\bigcup_{v \in S_G} V(H_v)\right)$, for some $S_G \subseteq V(G)$. Since $V(G) \subseteq S$, every vertex outside S belongs to some copy H_x with $x \notin S_G$. Such a vertex is adjacent to$x \in V(G) \subseteq S$, so Sdominates $G \circ H$.

Let $u \in V(G \circ H) \setminus S$. Then $u \in V(H_x)$for some $x \notin S_G$. The vertex u is adjacent to $x \in S$ and also to vertices in $H_x \setminus S$. Hence, the restrained condition holds.

Now let $a, b \in S$. If both vertices belong to $V(G)$, then any geodesic between them lies entirely in $V(G)$. If both belong to the same copy H_v, then either they are adjacent or connected through $v \in S$. If one vertex belongs to $V(G)$ and the other to some H_v, then they are connected through $v \in S$. Hence, every geodesic between vertices of S is contained in$S$, proving that S is weakly convex. Therefore, S is a weakly convex restrained dominating set of $G \circ H$.

Case 3: Suppose that S satisfies condition (iii). Then $S = V(G) \cup \left(\bigcup_{v \in S_G} S_v\right)$ where $S_G \subseteq V(G)$ and $S_v \subseteq V(H_v)$. Since $V(G) \subseteq S$, every vertex outside Sbelongs to some copy H_v and is adjacent to $v \in S$. Thus, S dominates $G \circ H$.

Let $u \in V(G \circ H) \setminus S$. Then $u \in V(H_v) \setminus S_v$ for some $v \in S_G$, or$u \in V(H_x)$ for some $x \notin S_G$. In either case, u is adjacent to a vertex in S and also to vertices outside S in its corresponding copy. Hence, the restrained condition holds.

For any two vertices in S, geodesics are contained in S because every path involving a copy H_v passes through the vertex $v \in V(G) \subseteq \setminus S$. Hence, S is weakly convex. Therefore, S is a weakly convex restrained dominating set of $G \circ H$.

Case 4: Suppose that S satisfies condition (iv). Then $S = V(G) \cup \left(\bigcup_{v \in S_G} V(H_v)\right) \cup \left(\bigcup_{x \in V(G) \setminus S_G} S_x\right)$, where $S_G \subseteq V(G)$ and $S_x \subseteq V(H_x)$ for each$x \in V(G) \setminus S_G$. Again, since $V(G) \subseteq S$, every vertex outside S belongs to some copy H_x and is adjacent to$x \in S$. Hence, S dominates $G \circ H$.

Let $u \in V(G \circ H) \setminus S$. Then u belongs to some copy H_x and has a neighbor in S, namely $x \in V(G)$, and also neighbors outside S in the same copy. Thus, the restrained condition holds.

Finally, every geodesic between vertices of S remains entirely in S since all vertices of $V(G)$ belong to S and each copy H_v is attached through the vertex $v \in V(G)$. Hence, S is weakly convex. Therefore, S is a weakly convex restrained dominating set of $G \circ H$.

As a direct consequence of the preceding characterization, the weakly convex restrained domination number of the corona of two graphs is obtained.

Corollary 1.19: *Let G and H be nontrivial connected graphs. Then,* $\gamma_{wcr}(G \circ H) = |V(G)|$.

Proof: By the preceding theorem, every weakly convex restrained dominating set S of $G \circ H$ must contain $V(G)$. Hence, $|S| \geq |V(G)|$.

On the other hand, the set $S = V(G)$ satisfies condition (i) of the theorem, and hence is a weakly convex restrained dominating set of $G \circ H$. Therefore, $\gamma_{wcr}(G \circ H) = |V(G)|$.

III. Conclusion

This paper introduced the concept of weakly convex restrained domination in graphs and the corresponding parameter $\gamma_{wcr}(G)$. Exact values were determined for several graph classes, including cycles, complete graphs, fan graphs, wheel graphs, and complete bipartite graphs. In particular, it was shown that weakly convex restrained dominating sets do not exist for cycles C_n when $n \geq 7$. Moreover, characterizations of weakly convex restrained dominating sets in the join and corona of two nontrivial connected graphs were established, leading to explicit formulas for the weakly convex restrained domination number under these graph operations. These results contribute to domination theory by highlighting the interaction between domination, convexity, and graph structure.

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