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A New Class of Integral Involving Generalized Hypergeometric Function and the H-Function

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Abstract: The aim of this research paper is to evaluate an interesting integral involving generalized hypergeometric function and the H-function. The integral is evaluated with the help of an integral involving generalized hypergeometric function obtained recently by Kim et al. The integral is further used to evaluate an interesting summation formula involving H- function. A few interesting special cases have also been given.

Keywords: Generalized Hypergeometric function, H – function, integral

1. Introduction

The H-function introduced by Fox[2] and studied by Braaksma[1] will be defined and represented in the following manner:

$$H_{p,q}^{m,n}\left[z\middle|\begin{matrix}{}_1(a_j,e_j)_p\\ {}_1(b_j,f_j)_q\end{matrix}\right]=\frac{1}{2\pi i}\int_L \theta(s)z^s\,ds \tag{1.1}$$

where $\theta(s)$ is given by

$$\theta(s)=\frac{\prod_{j=1}^{m}\Gamma(b_j-f_js)\prod_{j=1}^{n}\Gamma(1-a_j+e_js)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+f_js)\prod_{j=n+1}^{p}(a_j-e_js)} \tag{1.2}$$

Also,

(i) $z\neq 0$

(ii) $i=\sqrt{-1}$

(iii) m,n,p,q are integers satisfying $0\leq m\leq q$, $0\leq n\leq p$(not both zero simultaneously)

(iv) An empty product is to be interpreted as unity.

(v) $a_j,\ j=1,\ldots\ldots,p;\ b_j,\ j=1,\ldots.q$ are complex numbers.

(vi) $e_j,\ j=1,\ldots\ldots,p;\ f_j,\ j=1\ldots\ldots,q$ are real positive numbers for standardized purposes.

(vii) L is a contour goes from $\sigma-i\infty$ to $\sigma+i\infty$ (σ real) so that all the poles of $\Gamma(b_j-f_js),\ j=1,2,\cdots m$, lie to the right of L and all the poles of $\Gamma(1-a_j+e_js),\ j=1,2,\cdots,n$ lie to the left of L.

Braaksma[1] has shown that the integral (1.1) converges absolutely if

$$\theta>0,\ |argz|<\frac{\theta\pi}{2},$$

where θ is given by

$$\theta=\sum_{j=1}^{m}f_j-\sum_{j=m+1}^{q}f_j+\sum_{j=1}^{n}e_j+\sum_{j=n+1}^{p}e_j \tag{1.3}$$

Also from Braaksma [1]

$$H_{p,q}^{m,n}[z]\sim O[z^{\alpha}]$$

for small values of z , where

$$\alpha=\min_{1\leq j\leq m} Re\left(\frac{b_j}{f_j}\right)$$

and

$$H_{p,q}^{m,n}[z] \sim O[z^{\beta}]$$

for large value of z , where

$$\beta = \max_{1\le j\le n} Re\left(\frac{a_j - 1}{e_j}\right)$$

For more detail about H-function, we refer [5].

2. Results Required

The following integral involving generalized hypergeometric function obtained recently by Kim et al. [4] will be required in our present investigation.

$$\int_0^1 x^{c-1}(1-x)^c \, {}_3F_2\left[\begin{matrix} a, & b, & d+1 \\ \frac{1}{2}(a+b+1), & & d \end{matrix} ; x\right] dx$$
$$= \frac{\pi\ \Gamma(c)\ 2^{-2c}\ \Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\ \Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\ \Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\ \Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\ \Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}$$
$$+ \left(\frac{2c-d}{d}\right)\frac{\pi\ \Gamma(c)\ 2^{-2c}\ \Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\ \Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a\right)\ \Gamma\left(\frac{1}{2}b\right)\ \Gamma\left(c-\frac{1}{2}a+1\right)\ \Gamma\left(c-\frac{1}{2}b+1\right)} \quad (2.1)$$

provided $Re(2c-a-b) > -1$ and $d \in \mathbb{C} \setminus \mathbb{Z}_0^-$.

3. Main Integral

The following interesting integral involving generalized hypergeometric function and the H-function will be evaluated.

$$\int_0^1 x^{c-1}(1-x)^c \, {}_3F_2\left[\begin{matrix} a, b, d+1 \\ \frac{1}{2}(a+b+1), d \end{matrix} ; x\right] H_{p,q}^{m,n}\left[z\, x^{\lambda}(1-x)^{\lambda} \left| \begin{matrix} {}_1(a_j, e_j)_p \\ {}_1(b_j, f_j)_q \end{matrix} \right.\right] dx$$

$$= C_1\ H_{p+2,\ q+2}^{m,\ n+2}\left[z\, 2^{-2\lambda} \left| \begin{matrix} (1-c,\lambda), \left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b-c,\lambda\right), {}_1(a_j, e_j)_p \\ {}_1(b_j, f_j)_q, \left(\frac{1}{2}+\frac{1}{2}a-c,\lambda\right), \left(\frac{1}{2}+\frac{1}{2}b-c,\lambda\right) \end{matrix} \right.\right]$$

$$+ C_2\ H_{p+3,\ q+3}^{m,\ n+3}\left[z\, 2^{-2\lambda} \left| \begin{matrix} (1-c,\lambda), \left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b-c,\lambda\right), (d-2c, 2\lambda), {}_1(a_j, e_j)_p \\ {}_1(b_j, f_j)_q, \left(\frac{1}{2}a-c,\lambda\right), \left(\frac{1}{2}b-c,\lambda\right), (1+d-2c, 2\lambda) \end{matrix} \right.\right] \quad (3.1)$$

provided $\lambda > 0$, $\mathrm{Re(c)} > 0$, $\min_{1\le j\le m} Re\left[c + \lambda\ \left(\frac{b_j}{f_j}\right)\right] > 0$, $d \neq 0, -1, -2, \cdots$, $\theta > 0$, $|arg z| < \frac{\theta\pi}{2}$ where θ is the same as given in (1.3) . The values of C_1 and C_2 are given by

$$C_1 = \frac{\pi\ 2^{-2c}\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)} \quad (3.2)$$

and

$$C_2 = \frac{\pi\ 2^{-2c}\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\mathrm{d}\ \Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)} \quad (3.3)$$

Proof : In order to establish our main integral formula (3.1) , we proceed as follows.

Denoting the left hand side of (3.1) by I, expressing the H-function by its definition using (1.1) , interchanging the order of integration (which is easily seen to be justified with the conditions given due to the absolute convergence of the integrals involved in the process), we have

$$I = \frac{1}{2\pi\iota}\int_L \theta(s) z^s \left\{ \int_0^1 x^{c+\lambda s-1}(1-x)^{c+\lambda s} \; {}_3F_2\left[\begin{matrix} a, b, d+1 \\ \frac{1}{2}(a+b+1), d \end{matrix} ; x\right] dx \right\} ds$$

Evaluating the above integral with the help of the known result (2.1), we have after some simplification

$$I = \frac{\pi 2^{-2c}\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)} \cdot \frac{1}{2\pi\iota}\int_L \theta(s) z^s \frac{2^{-2\lambda s}\,\Gamma(c+\lambda s)\,\Gamma\left(c+\lambda s-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(c+\lambda s-\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(c+\lambda s-\frac{1}{2}b+\frac{1}{2}\right)}\, ds$$
$$+ \frac{\pi\, 2^{-2c}\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{d\;\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)}$$
$$\times \frac{1}{2\pi\iota}\int_L \theta(s) z^s \frac{2^{-2\lambda s}\,\Gamma(c+\lambda s)\,\Gamma\left(c+\lambda s-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)\Gamma(2c-d+2\lambda s+1)}{\Gamma\left(c+\lambda s-\frac{1}{2}a+1\right)\Gamma\left(c+\lambda s-\frac{1}{2}b+1\right)\Gamma(2c-d+2\lambda s)}\, ds$$

Finally, using the definition of H-function (1.1), we easily arrive at the right-hand side of (3.1). This complete the proof of (3.1).

4. An application in obtaining a new summation formula

In this section, as an application, we shall establish the following interesting summation formula involving H – function :

$$\sum_{r=0}^{\infty} \frac{(a)_r (b)_r (d+1)_r}{\left(\frac{1}{2}(a+b+1)\right)_r (d)_r\, r!} H_{p+2,q+1}^{m,n+2}\left[z \left| \begin{matrix} (1-c-r,\lambda),(-c,\lambda),\, {}_1(a_j,e_j)_p \\ {}_1(b_j,f_j)_q,\quad (-2c-r,2\lambda) \end{matrix}\right.\right]$$
$$= C_1\, H_{p+2,q+2}^{m,n+2}\left[\frac{z}{2^{2\lambda}} \left| \begin{matrix} (1-c,\lambda),\left(\frac{1}{2}-c+\frac{1}{2}a+\frac{1}{2}b,\lambda\right),\, {}_1(a_j,e_j)_p \\ {}_1(b_j,f_j)_q,\left(\frac{1}{2}-c+\frac{1}{2}a,\lambda\right),\left(\frac{1}{2}-c+\frac{1}{2}b,\lambda\right) \end{matrix}\right.\right]$$
$$+ C_2\, H_{p+3,q+3}^{m,n+3}\left[\frac{z}{2^{2\lambda}} \left| \begin{matrix} (1-c,\lambda),\left(\frac{1}{2}-c+\frac{1}{2}a+\frac{1}{2}b,\lambda\right),(d-2c,2\lambda),\, {}_1(a_j,e_j)_p \\ {}_1(b_j,f_j)_q,\left(\frac{1}{2}a-c,\lambda\right),\left(\frac{1}{2}b-c,\lambda\right),(1+d-2c,2\lambda) \end{matrix}\right.\right] \qquad (4.1)$$

where C_1 and C_2 are the same as given in (3.2) and (3.3) respectively.

Proof : In order to establish the summation formula (4.1), first we shall establish the following integral.

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1} H_{p,\,q}^{m,\,n}\left[zx^\lambda (1-x)^\mu \left| \begin{matrix} {}_1(a_j,e_j)_p \\ {}_1(b_j,f_j)_q \end{matrix}\right.\right] dx$$

$$= H_{p+2,\,q+1}^{m,\,n+2}\left[z \left| \begin{matrix} (1-\alpha,\lambda),(1-\beta,\mu),\, {}_1(a_j,e_j)_p \\ {}_1(b_j,f_j)_q,\quad (1-\alpha-\beta,\quad \lambda+\mu) \end{matrix}\right.\right] \qquad (4.2)$$

provided $\lambda > 0$, $\mu > 0$, $\min_{1\le j\le m} Re\left[\alpha+\lambda\frac{b_j}{f_j}\right] > 0$, $\min_{1\le j\le m} Re\left[\beta+\mu\frac{b_j}{f_j}\right] > 0$, $\theta > 0$ and $|\arg z| < \frac{\theta\pi}{2}$, where θ is the same as given in (1.3).

In order to establish the result (4.2), denote the left hand side of (4.2) by J we have

$$J = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\ H_{p,\ q}^{m,\ n}\left[zx^{\lambda}\ (1-x)^{\mu}\ \middle|\ \begin{matrix} {}_1(a_j, e_j)_p \\ {}_1(b_j, f_j)_q \end{matrix}\right] dx$$

Now, express the H-function with the help of definition (1.1), change the order of integration, which is justified due to the absolute convergence of the integrals involved in the process, we have after some simplification

$$J = \frac{1}{2\pi i}\int_L \theta(s)\ z^s\ \left\{\int_0^1 x^{\alpha+\lambda s-1}\ (1-x)^{\beta+\mu s-1}\ dx\right\}\ ds$$

Evaluating the beta- integral, we have

$$J = \frac{1}{2\pi i}\int_L \theta(s)\ z^s\ \frac{\Gamma(\alpha+\lambda s)\ \Gamma(\beta+\mu s)}{\Gamma(\alpha+\beta+(\lambda+\mu)s\ )}\ ds$$

Finally, Interpreting the result with the help of the definition of the H-function, we easily arrive at the right-hand side of (4.2).

The complete the proof of (4.2).

Next, denote the left- hand side of (3.1) by I_1 we have,

$$I = \int_0^1 x^{c-1}(1-x)^c\ {}_3F_2\left[\begin{matrix} a, & b, & d+1 \\ \frac{1}{2}(a+b+1), & & d \end{matrix}\ ;\ x\right]\ H_{p,\ q}^{m,\ n}\left[zx^{\lambda}\ (1-x)^{\mu}\ \middle|\ \begin{matrix} {}_1(a_j, e_j)_p \\ {}_1(b_j, f_j)_q \end{matrix}\right] dx$$

Expressing ${}_3F_2$ as a series, we have after some simplification

$$I = \sum_{r=0}^{\infty}\frac{(a)_n(b)_n(d+1)_n}{\left(\frac{1}{2}(a+b+1)\right)_n\ (d)_n\ n!}\int_0^1 x^{c+r-1}(1-x)^c\ H_{p,\ q}^{m,\ n}\left[zx^{\lambda}\ (1-x)^{\mu}\ \middle|\ \begin{matrix} {}_1(a_j, e_j)_p \\ {}_1(b_j, f_j)_q \end{matrix}\right] dx$$

Finally, evaluating the integral with the help of the result (4.2), we have

$$I = \sum_{r=0}^{\infty}\frac{(a)_r(b)_r(d+1)_r}{\left(\frac{1}{2}(a+b+1)\right)_r\ (d)_r\ r!}\ H_{p+2,\ q+1}^{m,\ n+2}\left[z\ \middle|\ \begin{matrix} (1-c-r,\lambda), (-c,\lambda),\ {}_1(a_j, e_j)_p \\ {}_1(b_j, f_j)_q\ , \quad (-2c-r, 2\lambda) \end{matrix}\right] \tag{4.3}$$

Hence, the required summation formula (4.1) follows from equating the two integrals (3.1) and (4.3).

This completes the proof of the summation formula (4.1).

In particular, when $d = 2c$, we get the following result.

$$\sum_{r=0}^{\infty}\frac{(a)_r\ (b)_r\ (2c+1)_r}{\left(\frac{1}{2}(a+b+1)\right)_r\ (2c)_r\ r!}\ H_{p+2,\ q+1}^{m,\ n+2}\left[z\ \middle|\ \begin{matrix} (1-c-r,\lambda), (-c,\lambda),\ {}_1(a_j, e_j)_p \\ {}_1(b_j, f_j)_q\ , (-2c-r, 2\lambda) \end{matrix}\right]$$

$$= C_1\ H_{p+2,q+2}^{m,n+2}\left[\frac{z}{2^{2\lambda}}\ \middle|\ \begin{matrix} (1-c,\lambda), & \left(\frac{1}{2}-c+\frac{1}{2}a+\frac{1}{2}b,\lambda\right), & {}_1(a_j, e_j)_p \\ {}_1(b_j, f_j)_q, & \left(\frac{1}{2}-c+\frac{1}{2}a,\lambda\right), & \left(\frac{1}{2}-c+\frac{1}{2}b,\lambda\right) \end{matrix}\right]$$

$$+C_2\ H_{p+2,\ q+2}^{m,\ n+2}\left[\frac{z}{2^{2\lambda}}\ \middle|\ \begin{matrix} (1-c,\lambda), & \left(\frac{1}{2}-c+\frac{1}{2}a+\frac{1}{2}b,\lambda\right), (0, 2\lambda),\ {}_1(a_j, e_j)_p \\ {}_1(b_j, f_j)_q, \left(\frac{1}{2}a-c,\lambda\right), & \left(\frac{1}{2}b-c,\lambda\right), \quad (1, 2\lambda) \end{matrix}\right] \tag{4.4}$$

Similarly, other results can also be obtained.

5. Special Cases

In this section, we shall mention two very interesting special cases of our main summation formula (3.1).

(i) Let $b = -2r$ and replace a by $a + 2r$, where r is zero or a positive integer . In such case, one of the two terms on the right hand side of (3.1) will vanish and we get the following interesting result.

$$\int_0^1 x^{c-1}(1-x)^c \, {}_3F_2\left[\begin{matrix}-2r, a+2r, d+1 \\ \frac{1}{2}(a+1), d\end{matrix}; x\right] H_{p,q}^{m,n}\left[z\, x^\lambda (1-x)^\lambda \left|\begin{matrix} {}_1(a_j,e_j)_p \\ {}_1(b_j,f_j)_q\end{matrix}\right.\right] dx$$

$$= \frac{(-1)^r\sqrt{\pi}}{2^{2c}} \frac{\left(\frac{1}{2}\right)_r}{\left(\frac{1}{2}a+\frac{1}{2}\right)_r} H_{p+2,q+2}^{m,n+2}\left[\frac{z}{2^{2\lambda}}\left|\begin{matrix}(1-c,\lambda),\left(\frac{1}{2}+\frac{1}{2}a-c,\lambda\right),\ {}_1(a_j,e_j)_p \\ {}_1(b_j,f_j)_q,\left(\frac{1}{2}+\frac{1}{2}a+r-c,\lambda\right),\ \left(\frac{1}{2}-r-c,\lambda\right)\end{matrix}\right.\right] \quad (5.1)$$

provided that the condition easily obtainable from (3.1) are satisfied.

(ii) Let $b = -2r-1$ and replace a by $a+2r+1$, where r is zero or a positive integer. In such case, one of the two terms on the right-hand side of (3.1) will vanish and we get the following interesting result

$$\int_0^1 x^{c-1}(1-x)^c \, {}_3F_2\left[\begin{matrix}-2r-1, & a+2r+1, & d+1 \\ & \frac{1}{2}(a+1), & d\end{matrix}; x\right]$$

$$\times\ H_{p,q}^{m,n}\left[z\, x^\lambda (1-x)^\lambda \left|\begin{matrix} {}_1(a_j,e_j)_p \\ {}_1(b_j,f_j)_q\end{matrix}\right.\right] dx$$

$$= \frac{(-1)^{r-1}\sqrt{\pi}}{d\ 2^{2c+1}} \frac{\left(\frac{3}{2}\right)_r}{\left(\frac{1}{2}a+\frac{1}{2}\right)_r}$$

$$\times\ H_{p+3,q+3}^{m,n+3}\left[\frac{z}{2^{2\lambda}}\left|\begin{matrix}(1-c,\lambda), & \left(\frac{1}{2}a+\frac{1}{2}-c,\lambda\right), & (d-2c,2\lambda), & {}_1(a_j,e_j)_p \\ {}_1(b_j,f_j)_q,\left(\frac{1}{2}+\frac{1}{2}a+r-c,\lambda\right), & \left(-\frac{1}{2}-r-c,\lambda\right), & (1+d-2c,2\lambda)\end{matrix}\right.\right] \quad (5.2)$$

Similarly, other results can be obtained.

Since H-function is the most general function of one variable which include as special cases, Meijer's G-function, MacRobert's E-function , Wright's generalized hypergeometric function , Generalized hypergeometric function ${}_pF_q$, Whittaker function, Mittag - Leffler function and almost all elementary function, so from our generalized summation formulas , a large number of interesting special cases can we obtained. But we shall not record them due to the lack of space.

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