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ON WEAK AND STRONG INDEXERS OF GRAPHS

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Abstract: A (p,q)-graph G is said to be strongly k-indexable, if its vertices can be assigned distinct nonnegative integers 0,1,2,…,p-1 so that the values of the edges, obtained as the sum of the numbers assigned to their end vertices can be arranged in the arithmetic progression $k,k+1,k+2,\ldots k+(q-1)$. In this paper we introduce and study weakly indexable graphs: a graph G is said to be weakly k-indexable, if its vertices can be assigned distinct nonnegative integers0,1,2,…,p-1 so that the values of the edges, obtained as the sum of the numbers assigned to their end vertices form the multiset of numbers $k,k+1,k+2,\ldots,k+t$where t is a positive integer less than q-1. In this paper, we obtain some necessary conditions on weakly k-indexable graphs, strongly k-indexable graphs and investigate classes of graphs which admit weak or strong indexers.

Keywords: Indexer, weak indexer, strong indexer.

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1. Introduction

For terminology and notation in graph theory we follow Harary [6] and West [9].

Given a graph $G = (V,E)$, the set N of nonnegative integers, a subset A of N and a commutative binary operation $*: N \times N \to N,$ every vertex function $f: V(G) \to A$ induces an edge function $f^*: E(G) \to N$ such that $f^*(uv) = *(f(u), f(v)), f(u) * f(v), \forall uv \in E(G).$ Often it is of interest to determine the vertex functions $f$having a specified property P such that the induced edge functions f^* have a specified property Q, where P and Qneed not necessarily be distinct. In this paper, we are interested in the study of vertex functions f, for which the induced edge function f^* defined as $f^*(uv) = f(u) + f(v), \forall uv \in E(G).$ Such vertex functions are said to be additive and henceforth this particular induced map $f^*$of f will be denoted f^+. A vertex function f is said to be an additive labeling if both$f$ and f^+ are injective.

We adopt the following notations throughout this paper.

$$f(G) = \{f(u): u \in V(G)\}$$

$$f^+(G) = \{f^+(e): e \in E(G)\}$$

Acharya and Hegde [2,7] have introduced the concepts of indexable and strongly indexable graphs: A graph $G = (V,E)$ is said to be (k,d)-indexable if it admits a (k,d)-indexer, namely, a bijection $f: V(G) \to \{0,1,2,\ldots,p-1\}$ such that $f^+(G) \subseteq \{k, k+d, k+2d, \ldots, k+(q-1)d\}$. When $k = d = 1$, f is called an indexer. If $f^+(G) = \{k, k+d, k+2d, \ldots, k+(q-1)d\}$, then f is called a strong (k,d)-indexer. If $d = 1$, then a strong (k,d)-indexer $f$is called a strong k-indexer and G is said to be strongly k-indexable if it admits such an indexer for some k. If $k = d = 1$, then f is called a strong indexer and G is said to be strongly indexable if it admits such an indexer. An additive labeling f of a graph G is said to be an indexable labeling if $f: V(G) \to \{0,1,2,\ldots,p-1\}$ such that the values in $f^+(G)$ are all distinct. A graph which admits such a labeling is called an indexable graph.

In this paper we introduce the concept of weak indexer: A (k,d) indexer $f$is called a weak (k,d)-indexer, if edge values form the multiset $M(G) = \{k, k+d, k+2d, \ldots, k+td\}$, where t is a positive integer <q-1. A graph G which admits such an indexer for some k and d, is called a weakly (k,d)-indexable graph. If $d = 1$, then f is called a weak k- indexer. A graph which admits such an indexer f for some k, is called a weakly k-indexable graph. If $k = d = 1$, then f is called a weak indexer. A graph which admits such an indexer$f$ is called a weakly indexable graph. If $t = q$-1, then f becomes a strong (k,d)-indexer. In this paper, we obtain some necessary conditions on weakly k-indexable graphs, strongly k-indexable graphs and investigate classes of graphs which admit weak or strong indexers.

One can observe that for a weakly k-indexable graph G,

$$k \leq f^+(e) \leq k + 2p - 4.$$

It follows from the definition that, every strongly indexable graph is indexable, but not the converse. For example, one can see that C_4 with the vertex labels (0,1,3, 2) is an indexable graph but not strongly indexable (it is known from [8], that even cycles are not strongly indexable). Note that there are graphs, which are both strongly as well as weakly indexable. That means, they are label dependent. For example $K_2 + \overline{K_2}$ if vertices of the common edge are labeled 1 and 2 (or 0 and 1) and the other two vertices are labeled 0 and 3 (or 2 and 3), we get the required strongly (or weakly) indexable graph. Also the graphs C_3 and $K_{1,n}$ are examples for strongly indexable but not weakly indexable. The cycle C_4 with the labels (0,3,1,2) is an example for a weakly 2-indexable graph, which is not strongly k-indexable. The disconnected graph $2C_3$ with the labels (0,1,2), (3,4,5) is an example for an indexable graph which is neither strongly indexable nor weakly indexable. Therefore, it is interesting to study weakly indexable graphs and strongly indexable graphs.

Theorem 1[1]. For any graph $G=(V,E)$ and for any additive vertex function $f: V(G) \to N$,

$$\sum_{e \in E(G)} f^+(e) = \sum_{u \in V(G)} f(u)d(u) \qquad (1)$$

Theorem 2 [2] . For any (k,d)-indexable (p,q)-graph G,

$$q \leq \frac{(2p - 3 - k + d)}{d} \qquad (2)$$

If $d = 1$, then $k + q - 1 \leq 2p - 3.$

We recall the following definitions.

Definition 1 [4]. A complete r-partite graph is obtained by partitioning the vertex set into r sets and joining two vertices if and only if they lie in different sets. If all of these sets have size k , then the resulting graph is denoted by $K_r\,(k)$.

Definition 2. A cycle with n-pendant edges attached at each vertex is called the n-crown, denoted as $C_m \Theta \overline{K_n}.$

Definition 3. A helm is obtained from a wheel ($W_n = C_n + K_1$) by attaching pendant edges to each of the rim vertices of the cycle.

Definition 4. The generalized closed helm $CH\,(t,n)$, $t \geq 2,\ n \geq 3$ is obtained from a helm by joining the pendant vertices to form a cycle such that it contains t cycles and n vertices.

Definition 5. The generalized wheel is defined as $W_{m,n} = C_n + K_m$.

2. Weakly Indexable Graphs

In this section, we study the structural properties of weakly k-indexable graphs and the graphs admitting such a labeling.

Lemma 1. For a connected weakly k-indexable (p,q)-graph G,

$$1 \leq k \leq 2p - t - 3.$$

Proof. According to the definition, $f^+(G) = \{k, k+1, k+2, ..., k+t\}, t < q - 1.$ Since the maximum edge value is 2*p*-3,

$$k + t \leq 2p - 3$$

$$\Rightarrow \quad k \leq 2p - t - 3$$

Therefore

$$1 \leq k \leq 2p - t - 3.$$

Theorem 3 . Every connected weakly indexable graph contains a triangle.

Proof. Let $G = (V,E)$ be a weakly indexable (p,q)-graph with a weak indexer f . Let G contains at least four vertices. Observe that the edge values 1, 2, 2*p*-4 and 2*p*-3 can be obtained only by one choice of the pair (0,1), (0,2), (*p*-3,*p*-1) and (*p*- 2,*p*-1) respectively, i.e. 1= 0+1, 2= 0+2, 2*p*-4 = *p*-3 + *p*-1, 2*p*-3 = *p*-2 + *p*-1. But the edge values 3,4,…,2*p*-5 can be obtained by more than one choice of a pair (*a*,*b*). Since *G* is weakly indexable, at least one of the values 3, 4,…,2*p*-5 recurs. Without loss of generality, we can assume that either 3 or 2*p*-5 recurs.

Suppose 3 occurs twice. That means the pair (0,3) and (1,2) are joined by lines. Then, it follows that 0,1,2 forms a triangle.

On the other hand, if 2*p*-5 occurs twice. That means the pairs (*p*-4,*p*-1) and (*p*-3,*p*-2) are joined by lines. But, then (0, *p*-4,*p*-1) and (0,*p*-3,*p*-2) form triangles. Therefore, every connected weakly indexable graph contains a triangle. Hence the proof.

From the above theorem, it follows that the class of connected triangle-free graphs are not weakly indexable. This infinite class contains all bipartite graphs, cycles etc. Hence, if a unicyclic graph is weakly indexable, then its unique cycle must be a triangle.

Corollary 3.1. If *G* is a connected weakly indexable (p,q)-graph, then $4 \leq p \leq q$.

Proof. From the above theorem, it can be observed that every weakly indexable graph must contain at least four vertices. Since a tree is not weakly indexable, $q \geq p$ so that $4 \leq p \leq q$.

Corollary 3.2. If G is a weakly indexable graph with a triangle, then any weak indexer of *G* must assign 0 to a vertex of a triangle in *G*.

Theorem 4 . If $K_{a,b},\ 2 \leq a \leq b$ is weakly *k*-indexable, then $k \leq b$.

Proof. Let $V_1 = \{u_1, u_2, ..., u_a\}$ and V_2 = $V_2 = \{v_1, v_2, ..., v_b\}$ be the bipartition of$K_{a,b}$, where $|A| = a,\ |B| = b.$ Note that f ($K_{a,b}$) contains the numbers from 0 to $a + b$-1. Suppose $K_{a,b},\ 2 \leq a \leq b$ is weakly *k*-indexable for $k > b$. Without loss of generality, assign the number 0 to a vertex of V_2. Then the numbers 1,2,…,*b* are to be assigned to the vertices of V_2 . Since there are only *b*-vertices in V_2 , the number *b* has to be assigned to a vertex of V_1 . This implies an induced edge value *b*, which is a contradiction to our assumption that $k >$ b. Thus, *k* cannot be greater than *b*. Hence if $K_{a,b},\ 2 \leq a \leq b$ is weakly k-indexable,then $k \leq b.$

Theorem 5 . If *G* is an *r*-regular weakly *k*-indexable ($k > 1$) (p,q)-graph, then

$$\left\lceil \frac{1\{rp(4p - rp - 2) + 16\}}{4rp} \right\rceil \leq k \leq \left\lfloor \frac{1(rp^2 - 2rp + 2)}{rp} \right\rfloor.$$

Proof. Let $G = (V,E)$ be an *r*-regular weakly $k(> 1)$-indexable (p,q)-graph. Without loss of generality, we can assume that either *k*+1 or *k*+*q*-3 recurs. Suppose (*k*+1) occurs (*q*-1) times, i.e. $f^+(G) = \{k, k+1, k+1, ...q-1\ times\}$. Using equation (1), we get

$$\sum_{i=0}^{p-1} r\ f(u) \geq k+(k+1)(q-1)$$

$$r\sum_{i=0}^{p-1} i \geq kq+q-1$$

$$\frac{rp(p-1)}{2} \geq kq+q-1$$

But, for an r-regular graph, $q=\frac{rp}{2}$

$$\frac{rp(p-1)}{2} \geq \frac{krp}{2}+\frac{rp}{2}-1$$

$$\Rightarrow \quad rp^2-2rp+2 \geq krp$$

$$\Rightarrow \quad krp \leq rp^2-2rp+2,$$

From which we get

$$k \leq \left\lfloor \frac{1(rp^2-2rp+2)}{rp} \right\rfloor. \qquad (3)$$

If, on the other hand, $k+q$ -3 occurs twice, then from equation (1), we get

$$\frac{rp(p-1)}{2} \leq k+k+1+k+2+\ldots+k+q-4+k+q-3+k+q-3+k+q-2$$

$$\frac{rp(p-1)}{2} \leq q^2+2kq-q-4$$

Using $q=\frac{rp}{2}$, we get $\frac{rp^2-rp}{2} \leq \frac{(rp)^2}{4}-\frac{rp}{2}+krp-4$

$$4rp^2-r^2p^2-2rp+16 \leq 4krp$$

$$rp(4p-rp-2)+16 \leq 4krp$$

From, which we get $\quad k \geq \left\lceil \frac{1\{rp(4p-rp-2)+16\}}{4rp} \right\rceil \qquad (4)$

From equation (3) and (4), we get

$$\left\lceil \frac{1\{rp(4p-rp-2)+16\}}{4rp} \right\rceil \leq k \leq \left\lfloor \frac{1(rp^2-2rp+2)}{rp} \right\rfloor.$$

This completes the proof.

For example: The 3-regular graph as shown in Figure 1 is both weakly 2-indexable (see Fig.1 a) as well as weakly 3-indexable (see Fig.1b)). But one can verify that it is not weakly 4-indexable.

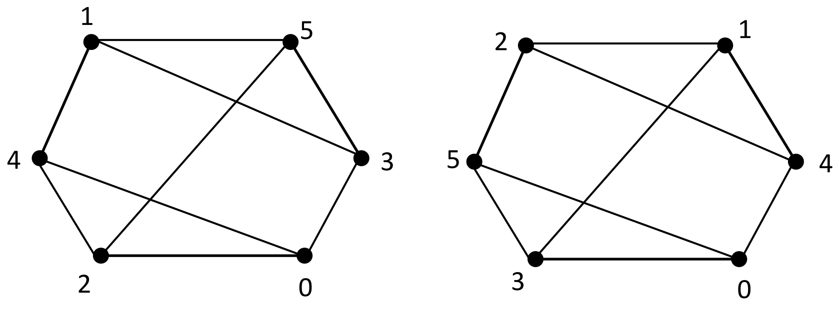

Figure 1: Weakly 2-indexable and weakly 3-indexable labelings.

Theorem 6 . The complete graph K_p, $p > 3$ is weakly indexable.

Proof. Denote the vertex set of the complete graph of order p; K_p as $V(K_p) = \{v_i : 0 \leq i \leq p-1\}$.
Define,

$$f(v_i) = i,\ 0 \leq i \leq p-1.$$

Then, one can easily observe that when p is even, the edge values (1, 2),(3, 4),(5, 6),…, (p -3, p- 2), (p-1, p); (p+1, p+2),…, (2p-6,2p-5), (2p-4; 2p-3) occur respectively (1, 1),(2, 2),(3, 3),…, $(\frac{p-2}{2}, \frac{p-2}{2})$, $(\frac{p}{2}, \frac{p-2}{2})$, $(\frac{p-2}{2}, \frac{p-4}{2})$, …(2,2),(1,1) times. when p is odd, the edge values (1,2),(3, 4),(5, 6),…, (p- 1, p),…, (2p -6,2p -5), (2p -4; 2p-3) occur respectively (1, 1),(2, 2),(3, 3),…, $(\frac{p-1}{2}, \frac{p-1}{2})$, …,(2,2),(1,1) times. Note that $f^+(K_p)$ contains edge values from 1 to 2p-3. Hence K_p is weakly indexable.

Next, we give a method to recursively enlarge a weakly indexable graph G to a weakly indexable graph H of higher order.

Denote the vertex set of the complete graph of order p; K_p as $V(K_p) = \{v_i : 0 \leq i \leq p-1\}$. Define the function f as above. Introduce $\overline{K_t}, t \geq 1$ new vertices and join them to each vertex of K_p by single new lines. Assign the values p+t-1 to the newly introduced vertices in a one-to-one manner. One can see that the resulting graph is weakly indexable with edge values from 1 to 2p + t-2.

Theorem 7 . For any integer $t \geq 2$, $r > 2$, the complete r-partite graph $K_r(t)$ is weakly k-indexable, where $k = t$.

Proof. Note that the complete r-partite graph for t=1, is nothing but the complete graph K_r , which is weakly indexable for $r > 3$. Let $r > 2, t \geq 2$. Note that $K_r(t)$ contains rt vertices and $\overset{r}{\underset{2}{C}}.t^2$ edges. Let $A_1 = \{u_{1,1}, u_{1,2}, ..., u_{1,t}\}$, $A_2 = \{u_{2,1}, u_{2,2}, ..., u_{2,t}\}$,..., $A_r = \{u_{r,1}, u_{r,2}, ..., ur, t\}$ be the r-partitions, each of size t. Define,

$$f(u_{i,j}) = (i-1)t + j - 1,\ 1 \leq i \leq r,\ 1 \leq j \leq t.$$

Then one can verify that $f^+(K_r(t))$ contains the edge values from t to $2rt$-t-2 where the middle term rt-1 occurs $\frac{rt}{2}$ times where r is even and $\left\lfloor \frac{r}{2} \right\rfloor .t$ times when r is odd.

Theorem 8. The helm H_n is weakly k-indexable, where $k = \frac{3n-1}{2}$ if n is odd $\frac{n}{2}$ and if n is even.

Proof.Let H_n be the helm. Note that H_n has 2n+1 vertices and 3n edges. Denote the rim vertices of H_n as $v_{1,1}, v_{1,2}, ..., v_{1,n}$; the pendant vertices adjacent to $v_{1,1}, v_{1,2}, ..., v_{1,n}$ as $v_{2,1}, v_{2,2}, ..., v_{2,n}$ and the centre as $v_{0,0}$. We consider two cases.

Case 1: n odd.

Define, $$f(v_{1,j}) = \begin{cases} 2n - \frac{(j+1)}{2}, j \ odd, 1 \le j \le n \\ \frac{3n-j-1}{2}, \ j \ even \ \ 2 \le j \le n-1 \end{cases}$$

$$f(v_{2,j}) = j-1, 1 \le j \le n$$

$$f(v_{0,0}) = 2n.$$

Then one can verify that $f^+(H_n) = \{\frac{3n-1}{2}, \frac{3n+1}{2}, ..., 4n-1\}$, where the edge values $3n, 3n+1, ..., \frac{7n-3}{2}$ occur two times. Therefore H_n is weakly k-indexable $k = \frac{3n-1}{2}$ for n is odd.

Case2 :n even.

Define, $$f(v_{1,j}) = \begin{cases} \frac{(j-1)}{2}, \ j \ odd, 1 \le j \le n-1 \\ \frac{n+j-2}{2}, \ j \ even \ \ 2 \le j \le n \end{cases}$$

$$f(v_{2,1}) = 2n$$

$$f(v_{2,j}) = \begin{cases} \frac{(3n+j-1)}{2}, j \ odd, 3 \le j \le n-1 \\ n + \frac{j}{2}, \ j \ even \ \ 2 \le j \le n \end{cases}$$

$$f(v_{0,0}) = n.$$

Then one can verify that $f^+(H_n) = \{\frac{n}{2}, \frac{n+2}{2}, ..., \frac{5n-2}{2}\}$, where the edge values $(n-1, n, ..., \frac{3n-4}{2})$ $(\frac{3n+2}{2}, \frac{3n+4}{2}, ..., 2n)$ occur two times. Therefore H_n is weakly k-indexable $k = \frac{n}{2}$ for n is even.

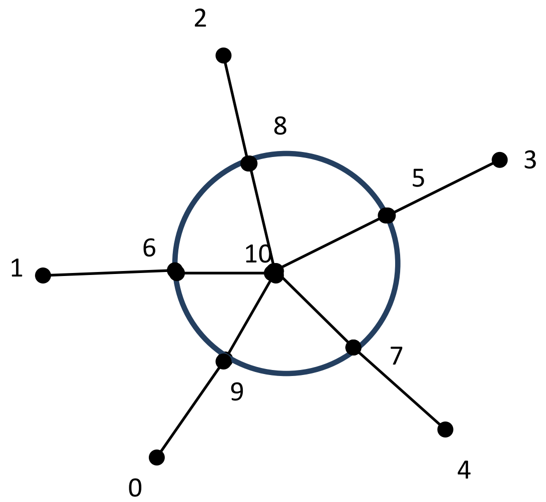

Fig.2: Weakly 7-indexable labeling of H_5.

Theorem 9. If *G* is a weakly (1,2)-indexable tree, then there exists a partition of $V(G)$ in to two parts V_1 and V_2 such that $|V_1|-|V_2|\leq 1$.

Proof. Suppose that *G* is a weakly (1,2)-indexable tree. Then $f^+(G)$ contains all odd numbers 1,3,…,2*q*-1.Without loss of generality,let $V_1=\{u: f(u)\equiv 0(\bmod 2)\}, V_2=\{v: f(v)\equiv 1(\bmod 2)\}$. Since $f(G)=\{0,1,\ldots,p-1\}$, the number of even numbers in this set is equal to zero or one more than the number of odd numbers. Hence $|V_1|-|V_2|\leq 1$.

But one can easily verify that the sub division graph of $K_{1,3}$ is not weakly (1,2)-indexable and hence the converse is not necessarily true.

Theorem 10. The disjoint union of cycles, ${}^mC_n$ is weakly (*k*,2)-indexable for

i) *m* even ≥ 2, *n* even ≥ 4

ii) *m* even ≥ 2, *n* odd ≥ 3

where $k=\frac{mn}{2}$ when *n* is even and $k=\frac{m(n-1)}{2}$ when *n* is odd.

Proof. Let ${}^mC_n$ denote the disjoint union of *m* cycles. Denote $V({}^mC_n)$ as $V_1\cup V_2\cup..\cup V_m$ where $V=\{v_i^1,v_i^2,\ldots,v_i^n: 1\leq i\leq m, 1\leq j\leq n\}$, v_i^j is the *j* th vertex of *i* th cycle. We consider two cases.

Define $f:V({}^mC_n)\rightarrow\{0,1,2,\ldots,mn-1\}$ as follows.

Case 1: *m* even *n* even ≥ 4.

Define, $f(v_i^{2j+1})=jm+i-1,\ 1\leq i\leq m,\ 0\leq j\leq\frac{n-2}{2}$

$f(v_i^{2j})=m(\frac{n}{2}+j-1),\ 1\leq i\leq m, 1\leq j\leq\frac{n}{2}$.

Then one can see that $f^+({}^mC_n)$ contains edge values from $\frac{mn}{2}$ to $\frac{3mn-4}{2}$ (which are even numbers)where all even numbers from *m*(*n*-1) to *m*(*n*+1)-2 occur at least twice. Hence ${}^mC_n$ *m* even ≥ 2, *n* even ≥ 4 is weakly (*k*,2)-indexable where $k = \frac{mn}{2}$.

Case 2: *m* even ≥ 2, *n* odd ≥ 3

Define, $f(v_i^{2j+1}) = jm + i - 1,\ 1 \leq i \leq m,\ 0 \leq j \leq \frac{n-1}{2}$

$$f(v_i^{2j}) = m(\frac{n+1}{2} + j - 1,\ 1 \leq i \leq m, 1 \leq j \leq \frac{n-1}{2}.$$

Then one can see that $f^+({}^mC_n)$ contains edge values from $\frac{m(n-1)}{2}$ to $\frac{3mn+m}{2} - 2$ (which are even numbers)where all even numbers from $\frac{m(n+1)}{2}$ to $\frac{m(3n-1)}{2} - 2$ occur twice. Hence ${}^mC_n$ *m* even ≥ 2, *n*odd≥ 3 is weakly (*k*,2)-indexable where $k = \frac{m(n-1)}{2}$.

Theorem 11.The disconnected graph $G = P_t \cup C_r$ is weakly *k*-indexable, where

i) $k = 2m + n + 2$, if $t = 2m+1,\ r = 2n+1, t < r$ and if $t = 2m+1,\ r = 2n, t < r$

ii) $k = 2m + n$ if $t = 2m,\ r = 2n,\ t < r$ and if $t = 2m,\ r = 2n+1, t < r$

iii) $k = m + 2n + 2$ if $t = 2m+1,\ r = 2n+1,\ t > r$ and if $t = 2m+1,\ r = 2n+1, t = r$

iv) $k = m + 2n + 1$, if $t = 2m+1,\ r = 2n, t > r$ and if $t = 2m,\ r = 2n+1, t > r$

v) $k = m + 2n$ if $t = 2m,\ r = 2n,\ t > r$ and if $t = 2m,\ r = 2n, t = r$.

Proof. Let P_t be a path on*t*-verticesand C_r be a cycle of length *r*. Denote the vertices of P_t as $u_1, u_2, ..., u_t$ and the vertices of C_r as $v_1, v_2, ..., v_r$. Note that $|V(G)| = t + r,\ \ |E(G)| = t + r - 1.$ We consider four cases.

Case 1: $t = 2m+1,\ r = 2n+1.$

Define $f : V(G) \to \{0,1,2,...,t+r-1\}$ by

$$f(u_{2i+1}) = i,\ 0 \leq i \leq m$$
$$f(u_{2i}) = m + 2n + i + 1,\ 1 \leq i \leq m$$
$$f(v_{2j-1}) = m + j,\ 1 \leq j \leq n+1$$
$$f(v_{2i}) = m + n + j + 1,\ 1 \leq j \leq n.$$

It can be easily verified that if *t*<*r*, then $f^+(G)$ contains edge values from 2*m*+*n*+2 to 2*m*+3*n*+2, where the values from *m*+2*n*+2 to 3*m*+2*n*+1 occur twice. if *t*=*r*, then $f^+(G)$ contains edge values from *m*+2*n*+2 to 2*m*+3*n*+2, where the values from *m*+2*n*+2 to 3*m*+2*n*+1 occur twice. if *t*>*r*, then $f^+(G)$ contains edge values from *m*+2*n*+2 to 3*m*+2*n*+1, where the values from 2*m*+*n*+2 to 2*m*+3*n*+2 occur twice.

Case 2: $t = 2m,\ r = 2n.$

Define $f : V(G) \to \{0,1,2,...,t+r-1\}$ by

$$f(u_{2i+1}) = i,\ 0 \leq i \leq m-1$$
$$f(u_{2i}) = m + 2n + i - 1,\ 1 \leq i \leq m$$
$$f(v_{2j-1}) = m + j - 1,\ 1 \leq j \leq n$$
$$f(v_{2i}) = m + n + j - 1,\ 1 \leq j \leq n.$$

It can be easily verified that if $t<r$, then $f^+(G)$ contains edge values from $2m+n$ to $2m+3n-2$, where the values from $m+2n$ to $3m+2n-2$ occur twice. If $t=r$, then $f^+(G)$ contains edge values from $m+2n$ to $3m+2n-2$, where all the values occur at least twice. if $t>r$, then $f^+(G)$ contains edge values from $m+2n$ to $3m+2n-2$, where the values from $2m+n$ to $2m+3n-2$ occur at least twice.

Case 3: $t=2m+1, r=2n$.

Define $f: V(G) \to \{0,1,2,...,t+r-1\}$ by

$f(u_{2i+1}) = i,\ 0 \le i \le m$

$f(u_{2i}) = m+2n+i+1,\ 1 \le i \le m$

$f(v_{2j-1}) = m+j,\ 1 \le j \le n$

$f(v_{2i}) = m+n+j,\ 1 \le j \le n.$

It can be easily verified that if $t<r$, then $f^+(G)$ contains edge values from $2m+n+2$ to $2m+3n$, where the values from $m+2n+1$ to $3m+2n$ occur at least twice. if $t>r$, then $f^+(G)$ contains edge values from $m+2n+1$ to $3m+2n$, where the values from $2m+n+2$ to $2m+3n$ occur at least twice.

Case 4: $t=2m, r=2n+1$.

Define $f: V(G) \to \{0,1,2,...,t+r-1\}$ by

$f(u_{2i+1}) = i,\ 0 \le i \le m-1$

$f(u_{2i}) = m+2n+i,\ 1 \le i \le m$

$f(v_{2j-1}) = m+j-1,\ 1 \le j \le n+1$

$f(v_{2i}) = m+n+j,\ 1 \le j \le n.$

It can be easily verified that if $t<r$, then $f^+(G)$ contains edge values from $2m+n$ to $2m+3n$, where the values from $m+2n+1$ to $3m+2n-1$ occur twice. if $t>r$, then $f^+(G)$ contains edge values from $m+2n+1$ to $3m+2n-1$, where the values from $2m+n$ to $2m+3n$ occur twice.

3. Strongly Indexable Graphs

In this section, we study the classes of graphs admitting strongly indexablelabelings.

Theorem 12. If an r-regular graph is strongly k-indexable, then $r \le 3$.

Proof.Let G be an r-regular strongly k-indexable graph with $r \ge 4$. Then $q = \frac{rp}{2} \ge 2p$. This is a contradiction to equation (2) for any $k \ge 1$. Hence if an r-regular graph is strongly k-indexable, then $r \le 3$.

Theorem 13[8]. A complete bipartite graph $K_{m,n}, m \le n$ is strongly k-indexable if and only if $m=1$.

Next, we obtain a necessary and sufficient condition for the join $G_1 + G_2$ of two graphs G_1 and G_2.

Theorem 14. The join of two graphs G_1 and G_2 is strongly indexable if and only if

i)at least one of G_1 and G_2 has exactly two vertices and

ii) $G_1 \cup G_2$ has exactly one edge.

Proof. Let G_1 be a (p_1, q_1)-graph and G_2 be a (p_2, q_2)-graph. Assume that $G_1 + G_2$ is a strongly indexable graph with $p_1 p_2 \ge 2.$ Let $|E(G_1 \cup G_2)| = m.$ Then $|V(G_1 + G_2)| = p_1 + p_2$ and $|E(G_1 + G_2)| = m + p_1 p_2$. Using equation (2),we get $m + p_1 p_2 \le 2(p_1 + p_2) - 3,$ which implies

$m+p_1p_2-2p_1-2p_2\le -3,$ so that $0\le (p_1-2)(p_2-2)\le 1-m$ (5)

Thus $m\le 1.$ Note that if m=0, then G_1+G_2 is a complete bipartite graph, which is not stronglyindexableby Theorem 13. So m must be 1.That means $|E(G_1\cup G_2)|=1.$ This in turn implies from equation (5) that $(p_1-2)(p_2-2)=0.$ Therefore either p_1 or p_2 must be 2.

For (ii), let $G=G_1+G_2$ be the join of the graphs G_1 and G_2. Suppose $2=p_1\le p_2$ and $|E(G_1\cup G_2)|=1.$ Then, we have two cases depending on the unique edge of $G_1\cup G_2$ belonging to G_1 or G_2. Hence the proof.

One can observe from Fig 3, that G is strongly indexable in both the cases.

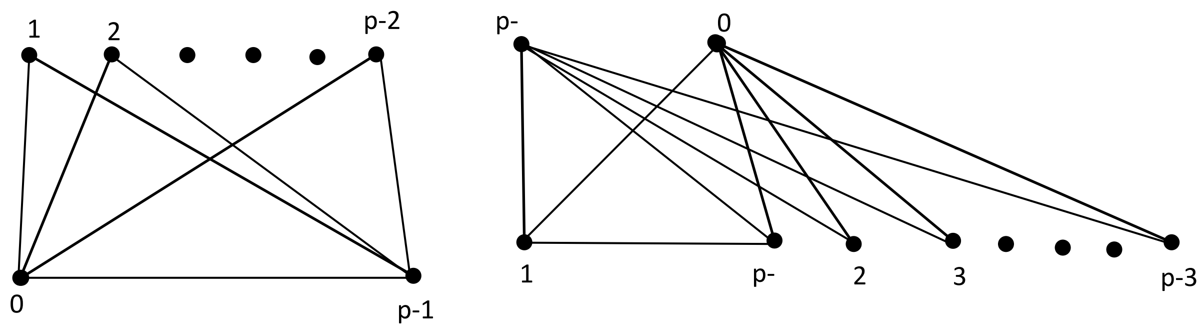

Fig. 3: Two different cases of G_1+G_2

Theorem 15. For any even $m\ge 4, n\ge 1,$ integer the n-crown $G=C_m\Theta\overline{K_n}$ is strongly k-indexable, where $k=\frac{m}{2}.$

Proof. Let $G=C_m\Theta\overline{K_n}$ be the n-crown. Assume that $m(even)\ge 4, n\ge 1.$ Denote the vertex set of G as $V(G)=\{u:1\le i\le m\}\cup\{v_{i,j}:1\le i\le m,\ 1\le j\le n\}$ and the edge set of G as $E(G)=\{u_i\,u_{i+1}:1\le i\le m-1\}\cup\{u_1u_m\}\cup\{u_1v_{i,j}:1\le i\le m, 1\le j\le n\}.$ Note $|V(G)|=|E(G)|=m(n+1).$ We proceed by the following eight cases.

Case 1:m=4.

Define the map

$f:V(G)\to\{0,1,2,\ldots,4n+3\}$ such that

$f(u_{2i-1})=i-1, i=1,2$

$f(u_{2i})=3i-1, i=1,2$

$f(v_{2i-1,1})=2(i+1), i=1,2$

$f(v_{2i,1})=11-4i, i=1,2$

$f(v_{i,j})=4(j+1)-i,\ 1\le i\le 4,\ 2\le j\le n.$

Then one can verify that f extends to a strongly k-indexable labeling of G for m=4.

Case 2: m=6.

Define the map $f:V(G)\to\{0,1,2,\ldots,6n+5\}$ such that $f(u_1)=8,\ f(u_2)=0,\ f(u_3)=3,\ f(u_4)=1,$

$f(u_5)=4,\ f(u_6)=2,\ f(v_{1,1})=5,\ f(v_{2,1})=7,\ f(v_{3,1})=6,\ f(v_{4,1})=11,\ f(v_{5,1})=10,$

$f(v_{6,1})=9,$

$$f(v_{i,j})=\begin{cases}5i+6j-5, 1\le i\le 2, 2\le j\le n\\ i+6j-2, 3\le i\le 6, 2\le j\le n.\end{cases}$$

Then one can verify that *f* extends to a strongly *k*-indexable labeling of *G* for *m*=6

Case 3:*m*=8.

Define the map $f:V(G)\to\{0,1,2,...,8n+7\}$ such that $f(u_1)=0,\ f(u_2)=4,\ f(u_3)=1,\ f(u_4)=5,$ $f(u_5)=2,\ f(u_6)=6,\ f(u_7)=3,\ f(u_8)=11,\ f(v_{1,1})=10,\ f(v_{2,1})=12,\ f(v_{3,1})=14,$ $f(v_{4,1})=13,\ f(v_{5,1})=15,\ f(v_{6,1})=7,\ f(v_{7,1})=9,\ f(v_{8,1})=8,$

$f(v_{i,j})=8(j+1)-i, 1\le i\le 8,\ 2\le j\le n.$

Then one can verify that *f* extends to a strongly *k*-indexable labeling of *G* for *m*=8.

Case 4:*m*=8*t*+2,where*t* is a positive integer.

Define the map $f:V(G)\to\{0,1,2,...,(8t+2)(n+1)-1\}$ such that

$$f(u_w)=\begin{cases}12t+2,\ if\ \ w=1\\ 4t+i-1,\ if\ \ w=2i-1,\ 2\le i\le 4t+1\\ i-1,\ if\ \ w=2i,\ 1\le i\le 4t+1\end{cases}$$

$$f(v_{w,1})=\begin{cases}8t+i,\ if\ \ w=2i-1,\ 1\le i\le 2t+2\\ 12t+1,\ if\ \ w=2\\ 12t+i+1,\ if\ \ w=2i,\ 2\le i\le 2t\\ 14t+2i+3,\ if\ \ w=4t+4i-2,\ 1\le i\le t\\ 14t-i+4,\ if\ \ w=4t+i+3,\ 1\le i\le 2\\ 10t+2i+1,\ if\ \ w=4t+4i+3,\ 1\le i\le t-1\\ 14t+2i+2,\ if\ \ w=4t+4i+4,\ 1\le i\le t-1\\ 10t+2i+2,\ if\ \ w=4t+4i+5,\ 1\le i\le t-1\\ 16t+2,\ if\ \ w=8t+2\end{cases}$$

and for $2\le j\le n,$ we have

$$f(v_{2i-1,j})=\begin{cases}2(4t+1)j+i-1,\ 1\le i\le 2t+1\\ 2(4t+1)j+i,\ 2t+2\le i\le 4t+1\end{cases}$$

$$f(v_{2i,j})=\begin{cases}(4t+1)(2j+1)+i-1,\ 2\le i\le 2t\\ (4t+1)(2j+1)+i-2,\ 2t+2\le i\le 4t+1\end{cases}$$

$$f(v_{2,j})=2(4t+1)(j+1)-1$$

$$f(v_{4t+2,j})=2(4t+1)j+2t+1.$$

Then one can verify that *f* extends to a strongly *k*-indexable labeling of *G* for *m*=8*t*+2.

Case 5:*m*=8*t*+4, where *t* is a positive integer.

Define the map $f:V(G)\to\{0,1,2,...,(8t+4)(n+1)-1\}$ such that

$$f(u_w)=\begin{cases} i-1, \text{ if } w=2i-1, 1\le i\le 4t+2 \\ 4t+i+1, \text{ if } w=2i, 1\le i\le 4t+1 \\ 12t+5, \text{ if } w=8t+4 \end{cases}$$

$$f(v_{w,1})=\begin{cases} 12t+4, \text{ if } w=1 \\ 16t-4i+7, \text{ if } w=4i-2, 1\le i\le t \\ 16t-4i+8, \text{ if } w=4i-1, 1\le i\le t \\ 16t-4i+9, \text{ if } w=4i, 1\le i\le t \\ 16t-4i+6, \text{ if } w=4i+1, 1\le i\prec t \\ 8t+3, \text{ if } w=4t+4i+2 \\ 16t+7, \text{ if } w=4t+3 \\ 16t+6 \text{ if } w=8t+3 \\ 8t+4, \text{ if } w=8t+4 \\ 12t-i+4, \text{ if } w=4t+i+3, 1\le i\le 4t-1 \end{cases}$$

$$f(v_{i,j})=4(2t+1)(j+1)-i, 1\le i\le 8t+4, 2\le j\le n.$$

Then one can verify that f extends to a strongly k-indexable labeling of G for $m=8t+4$.

Case 6: $m=8t+6$, where t is a positive integer.
Define the map $f:V(G)\rightarrow\{0,1,2,...,(8t+6)(n+1)-1\}$ such that

$$f(u_w)=\begin{cases} 12t+8, \text{ if } w=1 \\ 4t+i+1, \text{ if } w=2i-1, 2\le i\le 4t+3 \\ i-1, \text{ if } w=2i, 1\le i\le 4t+3 \end{cases}$$

$$f(v_{w,1})=\begin{cases} 8t+i+4, \text{ if } w=2i-1, 1\le i\le 2t+3 \\ 12t+7, \text{ if } w=2 \\ 12t+i+7, \text{ if } w=2i, 2\le i\le 2t+1 \\ 14t-2i+13, \text{ if } w=4t+3i+1, 1\le i\le 2 \\ 14t+2i+11, \text{ if } w=4t+4i+2, 1\le i\le t \\ 14t+2i+8, \text{ if } w=4t+4i+4, 1\le i\le t \\ 10t+2i+6, \text{ if } w=4t+4i+5, 1\le i\le t \\ 10t+2i+7, \text{ if } w=4t+4i+7, 1\le i\le t-1 \\ 16t+10, \text{ if } w=8t+6 \end{cases}$$

$$f(v_{w,j})=\begin{cases} 2(4t+3)j+i-1, \text{ if } w=2i-1, 1\le i\le 4t+3, 2\le j\le n \\ (4t+3)(2j+1)+i-1, \text{ if } w=2i, 1\le i\le 4t+3, 2\le j\le n. \end{cases}$$

Then one can verify that f extends to a strongly k-indexable labeling of G for $m=8t+6$.

Case 7: $m=16t$, where t is a positive integer.

Define the map
$f:V(G)\rightarrow\{0,1,2,...,16t(n+1)-1\}$

such that

$$f(u_w)=\begin{cases} i-1, \; if \;\; w=2i-1, \; 1\le i\le 8t \\ 8t+i-1, \; if \;\; w=2i, \; 1\le i\le 8t-1 \\ 24t-1, \; if \;\; w=16t \end{cases}$$

$$f(v_{1,1})=24t-2, \; f(v_{2,1})=16t+2, \; f(v_{3,1})=32t-2,$$

$$f(v_{w,1})=\begin{cases} 32t-2i, \; if \;\; w=2i-1, \; 3\le i\le 4t \\ 32t-2i+1, \; if \;\; w=2i, \; 2\le i\le 4t \\ 32t-3i+2, \; if \;\; w=8t+2i-1, \; 1\le i\le 2 \\ 24t-8i+4, \; if \;\; w=8t+8i-6, \; 1\le i\le t \\ 24t-8i+5, \; if \;\; w=8t+8i-4, \; 1\le i\le t \\ 24t-8i+3, \; if \;\; w=8t+8i-3, \; 1\le i\le t \\ 24t-8i-1, \; if \;\; w=8t+8i-2, \; 1\le i\le t \\ 24t+8i+1, \; if \;\; w=8t+8i-1, \; 1\le i\le t \\ 24t-8i, \; if \;\; w=8t+8i, \; 1\le i\le t \\ 24t-8i+2, \; if \;\; w=8t+8i+1, \; 1\le i\le t-1 \\ 24t-8i-2, \; if \;\; w=8t+8i+3, \; 1\le i\le t-1 \end{cases}$$

$$f(v_{i,j})=16t(j+1)-i, \; 1\le i\le 16t, \; 2\le j\le n.$$

Then one can verify that f extends to a strongly k-indexable labeling of G for m=16t.

Case 8: m=16t+8, where t is a positive integer.

Define the map

$$f:V(G)\to\{0,1,2,\ldots,(16t+8)(n+1)-1\}$$

such that

$$f(u_w)=\begin{cases} i-1, \; if \;\; w=2i-1, \; 1\le i\le 8t+4 \\ 8t+i+3, \; if \;\; w=2i, \; 1\le i\le 8t+3 \\ 24t+11, \; if \;\; w=16t+8 \end{cases}$$

$$f(v_{1,1})=24t+10, \; f(v_{2,1})=16t+10, \; f(v_{3,1})=32t+14,$$

$$f(v_{w,1})=\begin{cases}32t-2i+16, \text{ if } w=2i-1, 3\le i\le 4t+2\\ 32t-2i+17, \text{ if } w=2i, 2\le i\le 4t+2\\ 32t-3i+18, \text{ if } w=8t+2i+3, 1\le i\le 2\\ 24t-8i+15, \text{ if } w=8t+8i-2, 1\le i\le t+1\\ 24t-i+10, \text{ if } w=8t+i+7, 1\le i\le 2\\ 24t-8i+12, \text{ if } w=8t+8i+2, 1\le i\le t\\ 24t-8i+14, \text{ if } w=8t+8i+3, 1\le i\le t\\ 24t-8i+13, \text{ if } w=8t+8i+4, 1\le i\le t\\ 24t-8i+11, \text{ if } w=8t+8i+5, 1\le i\le t\\ 24t-8i+9, \text{ if } w=8t+8i+7, 1\le i\le t\\ 24t-8i+8, \text{ if } w=8t+8i+8, 1\le i\le t\\ 24t-8i+10, \text{ if } w=8t+8i+9, 1\le i\le t-1\end{cases}$$

$$f(v_{i,j})=(16t+8)(j+1)-i, 1\le i\le 16t+8, 2\le j\le n.$$

Then one can verify that *f* extends to a strongly *k*-indexable labeling of *G* for *m*=16*t*+*8*. Therefore G is strongly *k*- indexable for $k=\frac{m}{2}$.

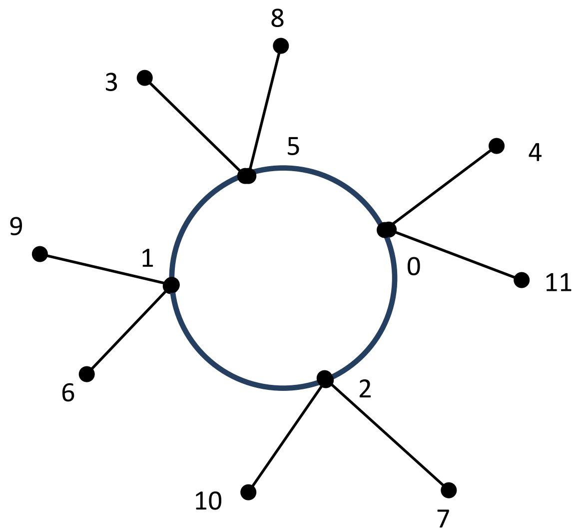

Fig.4: Strongly 2-indexable labeling of $C_4+\bar{K}_2$.

Theorem 16.For any odd integer $m\ge 3, n\ge 1,$ the *n*-crown $G=C_m\Theta\overline{K_n}$ is strongly *k*-indexable, where $k=\frac{m-1}{2}$.

Proof. Let $G=C_m\Theta\overline{K_n}$ be the *n*-crown. Let *m*(odd) $m\ge 3, n\ge 1.$ Denote the vertex set of *G* as $V(G)=\{u_i:1\le i\le m\}\cup\{v_{i,j}:1\le i\le m, 1\le j\le n\}$ and the edge set of *G* as $E(G)=\{u_iu_{i+1}:1\le i\le m-1\}\cup\{u_1u_m\}\cup\{u_iv_{i,j}:1\le i\le m, 1\le j\le n\}$. Notethat $|V(G)|=|E(G)|=$ m(n+1).

Define the map $f:V(G)\to\{0,1,2,...,m(n+1)-1\}$ such that

$$f(u_i)=\begin{cases}\frac{i-1}{2},\ for\ i\ odd,\ 1\le i\le m\\ \frac{n+i-1}{2},\ for\ i\ even\ 2\le i\le m-1\end{cases}$$

$$f(v_{i,j})=\begin{cases}nj+\frac{n+i}{2},\ for\ i\ odd,\ 1\le i\le m-2,\ 1\le j\le n\\ nj+\frac{i}{2},\ for\ i\ even\ 2\le i\le m-1,\ 1\le j\le n\end{cases}$$

$$f(v_{n,j})=nj,\ 1\le j\le n.$$

Then one can easily verify that the map *f* extends to a strongly *k*-indexable labeling of *G*, where $k=\frac{m-1}{2}$.

Conclusions

In this paper we obtained some classes of graphs which admit weak and strong indexers. Also we could obtain necessary conditions on weakly *k*-indexable and strongly *k*-indexable graphs.

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