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Perfect Doubly Connected Domination in the Join and Corona of Graphs

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Abstract: The perfect dominating set S is called a perfect doubly connected dominating set if it is doubly connected set in G. The minimum cardinality of a perfect doubly connected dominating set is called a perfect doubly connected domination number of G and is denoted by $\gamma_{pdc}(G)$. In this paper we investigate the concept and give some important results. Further, we characterize theperfect doubly connected dominating sets in the join and corona of two graphs.

Mathematics Subject Classification: 05C69

Keywords: dominating set, perfect dominating set, doubly connected dominating set, perfect doubly connected dominating set.

1. Introduction

Domination in graph was introduced by Claude Berge in 1958 and Oystein Ore in 1962 [1]. Some variants of domination in graphs are found in [2-8, 10]. One type of domination in graphs is the perfect domination. This was introduced by Cockayne et.al [11] in the paper "Perfect domination in graphs". The doubly connected domination is found in the paper "On the doubly connected domination number of a graph" by Cyman, Lemanska, and Raczek [12] and in the paper "Doubly connected domination in the corona and lexicographic product of graphs" by Arriola, and Canoy [13]and its variant [14]. For the general concepts, readers may refer to [15].

Let $G = (V(E), E(G))$ be a connected simple graph. The domination number $\gamma(G)$ of G is the smallest cardinality of a dominating set of G. A graph G is connected if there is at least one path that connects every two vertices $x, y \in V(G)$, otherwise, G is disconnected. A component of a graph is a maximal connected subgraph. Clearly, if a graph has only one component, then it is connected, otherwise it is disconnected. A dominating set $S \subseteq V(G)$ is called a connected dominating set of G if the subgraph $\langle S \rangle$ induced by S is connected. The connected domination number of G, denoted by$\gamma_c(G)$, is the smallest cardinality of a connected dominating set of G. A connected dominating set of cardinality $\gamma_c(G)$ is called aγ_c-set of G. A set $S \subseteq V(G)$ is a doubly connected dominating set if it is dominating and both $\langle S \rangle$ and $\langle V(G) \setminus S \rangle$ are connected. The doubly connected domination number of G, denoted by$\gamma_{cc}(G)$, is the smallest cardinality of a doubly connected dominating set S of G. A doubly connected dominating set of cardinality$\gamma_{cc}(G)$ is called aγ_{cc}-set of G. A dominating set $S \subseteq V(G)$ is called a perfect dominating set of G if each $u \in V(G) \setminus S$ is dominated by exactly one element of S. The perfect domination number of G, denoted by $\gamma_p(G)$, is the minimum cardinality of a perfect dominating set of G. A set $S \subseteq V(G)$ is a doubly connected dominating set if it is dominating and both $\langle S \rangle$ and $\langle V(G) \setminus S \rangle$ are connected. The perfect dominating set S is called an perfect doubly connected dominating set if it is doubly connected set in G. The minimum cardinality of a perfect doubly connected dominating set is called a perfect doubly connected domination number of G and is denoted by $\gamma_{pdc}(G)$. A perfect doubly connected dominating set of cardinality$\gamma_{pdc}(G)$ is called γ_{pdc}-set of G.

2. Results

From the definitions, the following result is immediate.

Remark 2.1 Let G be a connected nontrivial graph. Then $1 \leq \gamma_{pdc}(G) \leq n - 1$.

It is worth mentioning that the upper bound in Remark 2.1 is sharp. For example, $\gamma_{pdc}(P_n) = n - 1$for all $n \geq 2$. The next result says that the value of the parameter $\gamma_{pdc}(G)$ ranges over all positive integers less than or equal to $|V(G)| - 1$.

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Theorem 2.2 (Realization Problem)Given positive integers $k$and n such that $n \geq 2$ and $1 \leq k \leq n-1$, there exists a connected graph$G$with $|V(G)| = n$and $\gamma_{pdc}(G) = k$.

Proof: Consider the following cases:

Case1. Suppose that $1 \leq k < n-1$. Let$H$be nontrivial connected graph. Let $\mathcal{H} = \{H: \gamma(H) = 1\}$ with $|\mathcal{H}| = k$and let$n = \sum_{H\in\mathcal{H}} |V(H)|$. If $V(G) = \bigcup_{H\in\mathcal{H}} V(H)$such that$V(G') = \bigcup_{H\in\mathcal{H}}(V(H) \setminus \{v\})$where $\{v\}$ is a dominating set of$H$andG' is connected, then we obtain a particular graph of G (see figure 1).

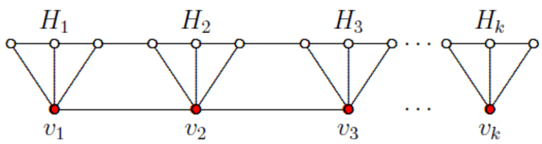

Figure 1: A graph G with $V(G) = \bigcup_{H\in\mathcal{H}} V(H)$.

The set $A = \{v_i: i = 1, 2, \ldots, k\}$ is aγ_{pdc}-$set$of G. Thus,$|V(G)| = |\bigcup_{H\in\mathcal{H}} V(H)| = \sum_{H\in\mathcal{H}} |V(H)| = nand\gamma_{pdc}(G) = k$ with $k = 1$ whenever the family $\mathcal{H}$ is trivial and$k < n-1$ whenever the family $\mathcal{H}$ is nontrivial.

Case2. Suppose that $k = n-1$. Let$G = P_n$. Then $|V(G)| = n$ and clearly,$\gamma_{pdc}(G) = n-1$.

This proves the assertion. ■

The next results characterized the perfect doubly connected dominating set with perfect doubly connected domination number of one.

Theorem 2.3 Let Gbe a connected nontrivial graph. Then$\gamma_{pdc}(G) = 1$ if and only if$G = K_1 + H$ where H is a connected graph.

Proof: Suppose that $\gamma_{pdc}(G) = 1$. Let $S = V(K_1)$ be aγ_{pdc}-set in G. Then $\langle V(G) \setminus S\rangle$ is a connected graph. Since$G$ is nontrivial,$S \neq V(G)$. Let $H = \langle V(G) \setminus S\rangle$. Then$G = K_1 + H$, where$H$ is a connected graph.

For the converse, suppose that $G = K_1 + H$, where H is a connected graph. Let $S = V(K_1)$. Then S is a perfect dominating set of G. Moreover, since $\langle V(G) \setminus S\rangle = H$ is a connected graph, it follows that S is a doubly connected dominating set of G. Thus, S is a perfect doubly connected dominating set of G. Hence, $\gamma_{pdc}(G) = 1$. ■

The next result is a quick consequence of Theorem 2.3.

Corollary 2.4 Let$F_n = K_1 + P_n$, $W_n = K_1 + C_n$, and K_n is a complete nontrivial graph. Then

$(i)\gamma_{pdc}(F_n) = 1$,
(ii) $\gamma_{pdc}(W_n) = 1$, and

$$(iii)\gamma_{pdc}(K_n) = 1.$$

The next results characterized the perfect doubly connected dominating set with perfect doubly connected domination number of two.

Theorem 2.5 Let Gbe a connected graph of order$n \geq 3$. Then$\gamma_{pdc}(G) = 2$ if and only if there exist vertices x and $y$such that$N_G[x] \cap N_G[y] = \{x, y\}$ is a dominating set and a subgraph induced by$V(G) \setminus \{x, y\}$ is connected.

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Proof: Suppose that $\gamma_{pdc}(G) = 2$. Let $S = \{x, y\}$ be a γ_{pdc}-set in G. Since S is doubly connected dominating set, $xy \in E(G)$ and $\langle V(G) \setminus S \rangle$ is connected and S is a dominating set. Let $z \in N_G[x] \cap N_G[y]$. Then $z \in N_G[x]$ and $z \in N_G[y]$. Suppose that $z \neq x$. If $z \neq y$, then $z \in V(G) \setminus S$ such that $zx, zy \in E(G)$ contrary to our assumption that S is a perfect dominating set of G. Thus, $z = y$ and hence $z \in S$, that is, $N_G[x] \cap N_G[y] \subseteq S$. Similarly if $z \neq y$, then $z = x$ and hence $N_G[x] \cap N_G[y] \subseteq S$. Since $xy \in E(G)$, it follows that $x, y \in N_G[x]$ and $x, y \in N_G[y]$, that is, $S = \{x, y\} \subseteq N_G[x] \cap N_G[y]$. Therefore $N_G[x] \cap N_G[y] = \{x, y\}$.

For the converse, suppose that there exist vertices x and y such that $N_G[x] \cap N_G[y] = \{x, y\}$ is a dominating set. Then $xy \in E(G)$. Let $S = \{x, y\}$. Then S is connected dominating set. Since $V(G) \setminus \{x, y\}$ is connected, it follows that S is a doubly connected dominating set of G. Now, since $n \geq 3$, there exists $z \in V(G) \setminus S$. Let $z \in N_G[x] \setminus S$. Then $z \neq y$. Since $S = N_G[x] \cap N_G[y]$, $zy \notin E(G)$, that is, each $z \in V(G) \setminus S$ is dominated by exactly one element $x \in S$. Similarly, if $w \in N_G[y] \setminus S$, then each $w \in V(G) \setminus S$ is dominated by exactly one element $y \in S$. Thus, S is a perfect dominating set of G. Accordingly, S is a perfect doubly connected dominating set of G. Since $\{x\}$ or $\{y\}$ is not a dominating set of G, it follows that $S = \{x, y\}$ is a minimum perfect doubly connected dominating set of G. Hence, $\gamma_{pdc}(G) = 2$. ■

Corollary 2.6 Let $m \geq 2$ and $n \geq 2$. Then $\gamma_{pdc}(K_{m,n}) = 2$.

Proof: Suppose $m \geq 2$ and $n \geq 2$ and let $G = K_{m,n}$. Then G is a connected graph of order $m + n \geq 4$ and there exist vertices x and y such that $xy \in E(G)$. This implies that $\{x, y\} \subseteq N_G[x] \cap N_G[y]$. Let $z \in N_G[x] \cap N_G[y]$. Then $z \in N_G[x]$ and $z \in N_G[y]$. If $z \neq x$, then $zx \in E(G)$. Since $y, z \in N_G[x]$ and G is a complete bipartite with $z \in N_G[y]$, it follows that $z = y$ (otherwise $G \neq K_{m,n}$). Similarly, if $z \neq y$, then $z = x$. This implies that $z \in \{x, y\}$. Hence $N_G[x] \cap N_G[y] \subseteq \{x, y\}$. Thus, $\{x, y\} = N_G[x] \cap N_G[y]$. Clearly, $\{x, y\}$ is a dominating set of G. Since $m + n \geq 4$, let $z, w \in V(G) \setminus \{x, y\}$. Let $x \in V(\overline{K}_n)$ and $y \in V(\overline{K}_n)$ where $V(G) = V(\overline{K}_n) \cup V(\overline{K}_m)$. If $z \in V(\overline{K}_n)$ and $w \in V(\overline{K}_m)$ then $zw \in E(G)$, that is, $\langle V(G) \setminus \{x, y\} \rangle$ is connected. Without loss of generality, suppose that $z \in V(\overline{K}_n)$ and $w \in V(\overline{K}_n)$. Then $zw \neq E(G)$. Since $m \geq 2$, let $v \in V(\overline{K}_m) \setminus \{y\}$. Then $vz, vw \in E(G)$, that is, $\langle V(G) \setminus \{x, y\} \rangle$ is connected. Accordingly, $\gamma_{pdc}(K_{m,n}) = 2$ by Theorem 2.5. ■

The join of two graphs G and H is the graph $G + H$ with vertex-set $V(G + H) = V(G) \cup V(H)$ and edge-set $E(G + H) = E(G) \cup E(H) \cup \{uv: u \in V(G), v \in V(H)\}$.

The next result presents a characterization of perfect doubly connected dominating sets in the join of two connected graphs.

Theorem 2.7 Let G and H be nontrivial graphs. Then $S \subseteq V(G + H)$ is a perfect doubly connected dominating set of $G + H$ if and only if one of the following holds.

(i) $S = \{x\}$ is a dominating set of G.
(ii) $S = \{y\}$ is a dominating set of H.
(iii) $S = \{x\} \cup \{y\}$, where x is an isolated vertex of G and y is an isolated vertex of H.

Proof: Suppose that $S \subset V(G + H)$ is a perfect doubly connected dominating set in $G + H$. If $S \cap V(H) = \emptyset$, then $S \subset V(G)$. Suppose that $|S| \geq 2$. Let $x, y \in S$. Then there exists $u \in V(H)$, that is, $u \in V(G + H) \setminus S$ such that $xu, yu \in E(G + H)$. This contradict to our assumption that S is a perfect dominating set of $G + H$. Thus, $|S| \ngeq 2$ and hence $|S| = 1$. Let $S = \{x\}$. Then S is a dominating set of G. This proves statement (i). Similarly, if $S \subset V(H)$, then statement (ii) holds. Now, suppose that $S_G = S \cap V(G) \neq \emptyset$ and $S_H = S \cap V(H) \neq \emptyset$. Then, $S = S_G \cup S_H$, where $S_G \subset V(G)$ and $S_H \subset V(H)$. This implies that $|S_G| = 1$ and $|S_H| = 1$ by (i) and (ii) respectively (otherwise S is not a perfect dominating set). Let $S_G = \{x\}$ and $S_H = \{y\}$. Suppose that there exists $v \in V(G)$ such that $xv \in E(G)$ and hence $xv \in E(G + H)$. Then $yv \in E(G + H)$. Since $S = \{x, y\}$, it follows that S is not a perfect dominating set of $G + H$ contrary to our assumption. Thus, x must be an isolated vertex of G. Similarly, y must be an isolated vertex of H (otherwise S is not a perfect dominating set). This proves statement (iii).

For the converse, suppose that statement (i) holds. Then S is a connected and perfect dominating set in $G + H$. Since G and H are a nontrivial graphs, let $v \in V(G) \setminus S$ and $u, y \in V(H)$. Then $v, u, y \in V(G + H) \setminus S$ where $vu, vy \in E(G + H)$. Since v, u and y are arbitrary elements of $V(G + H) \setminus S$, it follows that $\langle V(G + H) \setminus S \rangle$ is connected. Hence $S = \{x\}$ is a perfect doubly connected dominating set of $G + H$. Similarly, if statement (ii) holds, then $S = \{y\}$ is a perfect doubly connected dominating set of $G + H$. Now, suppose that statement (iii) holds. Then, $S = \{x, y\}$ is a dominating set $G + H$. Since $xy \in E(G + H)$, it follows that S is connected dominating set of $G + H$. Since x is an isolated vertex of G and H is nontrivial, there exists $u \in V(H) \setminus \{y\}$ such that $xu \in E(G + H)$ and since y is an isolated vertex of H and G is nontrivial, there exists \$$v \in V(G) \setminus \{x\}$ such that $yv \in E(G + H)$ for all $v \in V(G)$. Since $u, v \in V(G + H) \setminus S$, it follows that for each vertex $w \in V(G + H) \setminus S$ there exist exactly one vertex $z \in S$ such that $wz \in E(G + H)$. Hence S is a perfect dominating set of G. Now, since $uv \in E(G + H)$ for all $u, v \in V(G + H) \setminus S$ where $v \in V(G) \setminus \{x\}$ and $u \in V(H) \setminus \{y\}$, it follows that $\langle V(G + H) \setminus S \rangle$ is connected. Hence S is an doubly connected dominating set of $G + H$. Accordingly, S is a perfect doubly connected dominating set of $G + H$. ■

The next result follows immediately from Theorem 2.7.

Corollary 2.8 Let G and H be nontrivial graphs. Then $\gamma_{pdc}(G + H) = \begin{cases} 1 & if \quad \gamma(G) = 1 \; or \; \gamma(H) = 1 \\ 2 & if \quad each\; G \; and \; H \; has \; isolated \; vertex \end{cases}$

Let G and H be graphs of order m and n, respectively. The corona of two graphs G and H is the graph $G \circ H$ obtained by taking one copy of G and m copies of H, and then joining the ith vertex of G to every vertex of the ith copy of H. The join of vertex v of G and a copy H^v of H in the corona of G and H is denoted by $v + H^v$.

The following result is needed for the characterization of the perfect doubly connected dominating sets of the corona of graphs.

Theorem 2.9[1] Let G be connected graph and H be any graph. Then a nonempty set $C \subset V(G \circ H)$ is a dcd-set of $G \circ H$ if and only if there exists a vertex v of G such that $C = V(G) \cup (\cup_{u \in V(G)\setminus\{v\}} V(H^u)) \cup (V(H^v) \setminus T^v)$ where $\langle T^v \rangle$ is a connected subgraph of H^v.

Theorem 2.10 Let G be connected graph and H be any graph. Then a nonempty proper subset S of $G \circ H$ is a perfect doubly connected dominating set of $G \circ H$ if and only if there exists a vertex v of G such that $S = V(G) \cup \left(\cup_{u \in V(G)\setminus\{v\}}\right) \cup (V(H^v) \setminus T^v)$
where $\langle T^v \rangle$ is a connected subgraph of H^v.

Proof: Suppose that a nonempty proper subset S of $G \circ H$ is a perfect doubly connected dominating set of $G \circ H$. Then by Theorem 2.9, there exists a vertex v of G such that $S = V(G) \cup (\cup_{u \in V(G)\setminus\{v\}} V(H^u)) \cup (V(H^v) \setminus T^v)$ where $\langle T^v \rangle$ is a connected subgraph of H^v.

For the converse, suppose that there exists a vertex v of G such that $S = V(G) \cup (\cup_{u \in V(G)\setminus\{v\}} V(H^u)) \cup (V(H^v) \setminus T^v)$ where $\langle T^v \rangle$ is a connected subgraph of H^v. Then by Theorem 2.9, S is a doubly connected dominating set of $G \circ H$. Further, since $V(G \circ H) \setminus S \neq \emptyset$, let $x \in V(G \circ H) \setminus S$. Then $x \in T^v$ for some $v \in V(G)$, that is $xv \in E(G \circ H)$. Since $V(G) \subset S$, $v \in S$. This implies that each $x \in V(G \circ H) \setminus S$ is dominated by exactly one vertex $v \in S$, and hence, S is a perfect dominating set of $G \circ H$. Accordingly, S is a perfect doubly connected dominating set of $G \circ H$. ■

Corollary 2.11 Let G be connected graph of order $n \geq 1$ and H be any graph of order m. Then $\gamma_{pdc}(G \circ H) = mn + n - r$ where $r = max\{|V(J)|: J$ is a connected subgraph of $H\}$. In particular, if H is connected, then $\gamma_{pdc}(G \circ H) = mn + n - m$.

Proof: Let J be a connected subgraph H with $|V(J)| = r$. Choose $v \in V(G)$ and $T^v \subseteq V(H^v)$ such that $T^v = V(J)$. Then $S = V(G) \cup (\cup_{u \in V(G)\setminus\{v\}} V(H^u)) \cup (V(H^v) \setminus T^v)$ is a perfect doubly connected dominating set of $G \circ H$ by Theorem 2.10. Hence, $\gamma_{pdc}(G \circ H) \leq |S| = mn + n - r$. Now, let S' be a γ_{pdc}-set of $G \circ H$. Then $S' = V(G) \cup (\cup_{u \in V(G)\setminus\{v\}} V(H^u)) \cup (V(H^v) \setminus T^v)$ for some connected subgraph $\langle T_v \rangle$ in H^v. Hence, $\gamma_{pdc}(G \circ H) = |S'| = mn + n - |T^v| \geq mn + n - r$. Therefore, $\gamma_{pdc}(G \circ H) = mn + n - r$. ■

3. Conclusion And Recommendations

In this paper, we introduced a new parameter of domination in graphs - the perfect doubly connected domination in graphs. The perfect doubly connected domination in the join and corona of two graphs were characterized. The exact perfect doubly connected domination number resulting from these binary operations of two graphs were computed. This study will pave a way to new research such bounds and other binary operations of two graphs. Other parameters involving perfect doubly connected domination in graphs may also be explored. Finally, the characterization of a doubly connected domination in graphs and its bounds is a promising extension of this study.

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