International Journal of Latest Engineering Research and Applications (IJLERA) ISSN: 2455-7137

Volume –05, Issue – 02, February 2020, PP – 09-13

Theorem on Alternating Series Involving Binomial Coefficients, Exponentiated Integers and the Factorial

Nazia Kanwal
Garrison Post Graduate College for Women, Lahore Cantt.

Sumaira Ajmal Khan
Lahore Garrison University, Phase 6, DHA Lahore

Amber Javed
Lahore Garrison University, Phase 6, DHA Lahore

Abstract: The expression for alternating series of some integers along with Binomial coefficients or Pascal's identity whose solution forms a notion of factorial n.
Keywords: Factorials, Pascal identity,Binomial coefficients, Alternating series.

Introduction:

The factorial is a quantity defined for all integers greater than or equal to zero.Factorials in addition to binomial coefficients are normally defined in a rather narrow structure. So-called Pascal's Triangle with the study of factorials has always been a focus of need so referring the reader to [5] for more information. Furthermore, a sequenceis an enumerated collection of objects in which repetitions are allowed and order does matter and the sum of the sequence to a certain number of terms is known as series.Some special types of series are Harmonic series, Arithmetic series, Geometric series, Alternating series etc. This paper develops a new generalized expression involving positive s well as negative integers, comprised of alternating series and binomial coefficients, that leads to simple factorial of a number.

Here are some basic concepts related to the topic.

Preliminary Results:

Definition 1: [5]
Let X be a nonempty set, then a function f:N→ X whose domain is a set of natural numbers, is called an infinite sequence in X. If the domain of f is the finite set of numbers{1, 2,3,…,n} then it is called a finite sequence in X.

Definition 2: [3]
Mathematical notation uses a symbol that compactly represents summation of many similar terms: the summation symbol, □ , an enlarged form of the upright capital Greek letter Sigma.
This is defined as:

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \ldots\ldots + a_{n-1} + a_n$$

where i represents the index of summation; a_i is an indexed variable representing each successive term in the series; m is the lower bound of summation, and n is the upper bound of summation.

Definition 3: [4]
"A combination is any selection of objects where the order of the objects is immaterial (of no concern).
Using the letter C for combination, we have "the number of combinations of "r" objects from "n" different objects as ${}^{n}C_{r}$.

In general, to permute "r" objects from n different objects, we could first select the "r" objects in ${}^{n}C_{r}$ ways and then arrange these 'r' objects in r! ways.
Hence:

$${}^{n}C_{r} = {}^{n}P_{r} \div r! = \frac{n!}{r!(n-r)!}$$ where n= the total number of elements in a set

r= the number of selected objects !\= factorial

Definition 4: [4]
Expressions such as (x+y), (1-x) and (a+b) which contain two terms each are binomial expressions with unit power.
Draw the following rules concerning the expansions of $(1+\mathrm{b})^{n}$.

a) There are (n+1) terms.
b) The power of ‘b’ starts with 0 and increases to ‘n’, i.e its powers are in ascending order.

A more direct way to get any binomial coefficient is to use the ${}^{n}C_{r}$ notation

when n=4, there are 5 terms:	T_1	T_2	T_3	T_4	T_5
their coefficients are:	1	4	6	4	1
using the ${}^{n}C_{r}$ notation,	${}^{4}C_{0}$	${}^{4}C_{1}$	${}^{4}C_{2}$	${}^{4}C_{3}$	${}^{4}C_{4}$

Pascal’s triangle is a triangle where there is a row of 1’s down each side. Each number inside the triangle number comes from adding together the two numbers above it. [2]
using this notation, the Pascal’s triangle becomes:

$${}^{1}C_{0} \quad {}^{1}C_{1}$$
$${}^{2}C_{0} \quad {}^{2}C_{1} \quad {}^{2}C_{2}$$
$${}^{3}C_{0} \quad {}^{3}C_{1} \quad {}^{3}C_{2} \quad {}^{3}C_{3}$$
$${}^{4}C_{0} \quad {}^{4}C_{1} \quad {}^{4}C_{2} \quad {}^{4}C_{3} \quad {}^{4}C_{4}$$
$${}^{5}C_{0} \quad {}^{5}C_{1} \quad {}^{5}C_{2} \quad {}^{5}C_{3} \quad {}^{5}C_{4} \quad {}^{5}C_{5}$$

Thus, the coefficient of the 3rd and the 5th terms in the expansion of $(1+b)^{5}$ are ${}^{5}C_{2}$ and ${}^{5}C_{4}$ respectively.

“ In the expansion of $(1+b)^{n}$, the coefficient of $T_{r+1} = {}^{n}C_{r}$. Hence the binomial expansion of $(1+b)^{n}$, where $n \in Z^{+}$, is given by

$$(1+b)^{n} = 1 + {}^{n}C_{1}b^{1} + {}^{n}C_{2}b^{2} + {}^{n}C_{3}b^{3} + \cdots + {}^{n}C_{r}b^{r} + \cdots + b^{n}$$

This result is known as binomial theorem.

Lemma 1:-[1]

$$T_{x}^{j} = \sum_{i=0}^{n} (-1)^{i} \, {}^{n}C_{i} \, (x-i)^{j} = 0 \qquad j = 0,1,2,3,\ldots,n-1 \qquad (1.1)$$

Lemma 2:-[1]

$$T_{n}^{0} = \sum_{i=0}^{n} (-1)^{i} \, {}^{n}C_{i} = 0 \qquad \forall\, n > 0$$

Theorem:-
An alternating series consisting of (n+1) terms yields n factorial, if the terms are the product of Pascal’s coefficients of any power n with the corresponding (n+1) consecutive decreasing numbers exponentiated with n.

$$T_x^n = \sum_{i=0}^{n} (-1)^i \; {}^nC_i \left(x-i\right)^n = n! \qquad \forall\, x < n,\; n \in N\,,\; x \in N$$

Proof:-Using mathematical induction

For $x = n$

$$T_n^n = \sum_{i=0}^{n} (-1)^i \; {}^nC_i \left(n-i\right)^n \tag{1.2}$$

$$T_n^n = n\sum_{i=0}^{n-1} (-1)^i \; {}^{n-1}C_i \left(n-1-i+1\right)^{n-1}$$

$$T_n^n = n\sum_{i=0}^{n-1} (-1)^i \; {}^{n-1}C_i \sum_{l=0}^{n-1} {}^{n-1}C_l \left(n-1-i\right)^{n-1-l}$$

$$T_n^n = n\begin{bmatrix} {}^{n-1}C_0 \sum_{i=0}^{n-1} (-1)^i \; {}^{n-1}C_i \left(n-1-i\right)^{n-1} + \\ {}^{n-1}C_1\left(0\right) + {}^{n-1}C_2\left(0\right) + \cdots + {}^{n-1}C_{n-1}\left(0\right) \end{bmatrix} \qquad \text{by Lemma 1}$$

$$T_n^n = n \sum_{i=0}^{n-1} (-1)^i \; {}^{n-1}C_i \left(n-1-i\right)^{n-1}$$

$$T_n^n = n\left(n-1\right)\sum_{i=0}^{n-2} (-1)^i \; {}^{n-2}C_i \left(n-2-i\right)^{n-2}$$

and so on generalizing

$$T_n^n = n\left(n-1\right)\left(n-2\right)\left(n-3\right)\cdots 2\sum_{i=0}^{1} (-1)^i \; {}^1C_i \left(1-i\right)^1$$

$$T_n^n = n!$$

For $x = n+1$

$$T_{n+1}^n = \sum_{i=0}^{n} (-1)^i \; \frac{n!}{i!\left(n-i\right)!}\frac{1}{\left(n+1-i\right)}\left(n+1-i\right)^{n+1}$$

$$T_{n+1}^n = \frac{1}{n+1}\sum_{i=0}^{n} (-1)^i \; {}^{n+1}C_i \left(n+1-i\right)^{n+1}$$

$$T_{n+1}^n = \frac{1}{n+1}\sum_{i=0}^{n+1} (-1)^i \; {}^{n+1}C_i \left(n+1-i\right)^{n+1}$$

$$T_{n+1}^n = \frac{1}{n+1}\left(n+1\right)!$$

$$T_{n+1}^n = n!$$

For $x = n+2$

$$T_{n+2}^{n} = \frac{1}{n+1}\sum_{i=0}^{n}(-1)^{i}\ {}^{n+1}C_i\,(n+2-i)^n\,(n+1-i)$$

$$T_{n+2}^{n} = \frac{1}{n+1}\sum_{i=0}^{n}(-1)^{i}\ {}^{n+1}C_i\,(n+2-i)^n\,(n+1-i)$$

$$T_{n+2}^{n} = \frac{1}{n+1}\sum_{i=0}^{n+1}(-1)^{i}\ {}^{n+1}C_i\,(n+2-i)^n\,(n+2-i-1)$$

$$T_{n+2}^{n} = \frac{1}{n+1}\left\{\sum_{i=0}^{n+1}(-1)^{i}\ {}^{n+1}C_i\,(n+2-i)^{n+1} - 0\right\}$$

Put $n+1=m$ within summation in the expression

$$T_{n+2}^{n} = \frac{1}{n+1}\left\{\sum_{i=0}^{m}(-1)^{i}\ {}^{m}C_i\,(m+1-i)^{m}\right\}$$

$$T_{n+2}^{n} = \frac{1}{n+1}m!$$

$$T_{n+2}^{n} = \frac{1}{n+1}(n+1)!$$

$$T_{n+2}^{n} = n!$$

Suppose equation holds for $x = n+k$.

Now for $x = n+k+1$

$$T_{n+k+1}^{n} = \sum_{i=0}^{n}(-1)^{i}\ {}^{n}C_i\,(n+k+1-i)^n$$

$$T_{n+k+1}^{n} = \frac{1}{n+1}\sum_{i=0}^{n+1}(-1)^{i}\ {}^{n+1}C_i\,(n+k+1-i)^n\,(n+1-i)$$

$$T_{n+k+1}^{n} = \frac{1}{n+1}\sum_{i=0}^{n+1}(-1)^{i}\ {}^{n+1}C_i\,(n+k+1-i)^n\,(n+k+1-i-k)$$

$$T_{n+k+1}^{n} = \frac{1}{n+1}\sum_{i=0}^{n+1}(-1)^{i}\ {}^{n+1}C_i\,(n+k+1-i)^{n+1}$$

Put $n+1=m$ within summation in the expression

$$T_{n+k+1}^{n} = \frac{1}{n+1}T_{m+k}^{m}$$

$$T_{n+k+1}^{n} = \frac{1}{n+1}m! \qquad \text{as it holds for } k$$

$$T_{n+k+1}^{n} = \frac{1}{n+1}(n+1)! \quad \text{back substitution}$$

$$T_{n+k+1}^{n} = n!$$

Hence, completes the induction.

Conclusion:

This can be applicable to a number of theory problems. There is much more to be discovered by moving this work forward.

References

[1]. Kanwal, N., & Khan, S. A. (2018). Theorem on alternating series involving positive numbers and the factorial. *International Journal of Advanced Research*, 53-57.

[2]. Kanwal, N., Khan, S. A., Rana, Q., & Javed, A. (2019). A unique theorem onalternating series to simplify complexities. *International Journal of Latest Engineering Research and Applications 4(12)*, 1-5.

[3]. Kenneth, H. R., & Michael, J. G. (1999). *Handbook of Discrete and Combinatorial Mathematics* (ISBN 0-8493-0149-1 ed.). CRC Press.

[4]. Thong, H. S., & Hiong, K. N. (n.d.). *Additional Mathematics* (Dip ed.).

[5]. Yousaf, S. M., Majeed, A., & Amin, M. (2007). *Mathematical Methods.* Lahore: Ilmi Kitab Khana.