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Research on the Interpolation System of the Conical Radial Basis Function Algorithm

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Abstract: This study investigates the interpolation system parameter-free conical radial basis function algorithm. The effective condition number (ECN) is considered. The main difference between ECN and the traditional condition numbers is in that the ECN takes into account the right-hand side vector to estimates system stability. Even the ECN is a superior criterion over the traditional condition number for some numerical methods, it is not obvious for the conical radial basis function algorithm. Numerical results show that the ECN is not roughly inversely proportional to the numerical accuracy.

Keywords: Effective condition number, radial basis functions, Chebyshev node, boundary value problem, partial differential equation

1. Introduction

The radial basis functions based (RBF-based) meshless collocation methods perform well for interpolating multidimensional scattered data during the past several decades [1, 2]. They have been used to deal with many problems governed by the partial differential equations [3-5].

The conical radial basis functions method is proposed Zhang et al [6]. The Chebyshev node generation is considered for the solution of boundary value problems governed by the partial differential equations. The collocation points are placed uniformly or quasi-uniformly in the physical domain of the boundary value problems in question. Three different simple Chebyshev-type schemes are employed to generate the collocation points. This scheme improves accuracy of the method with no additional computational cost. Several numerical experiments are given to show the validity of the newly-proposed method.

In this paper, we make the attempt to investigate the interpolation system parameter-free conical radial basis functions, where the effective condition number (ECN) is also considered to see the stability analysis of parameter-free conical radial basis functions.

2. The Conical Radial Basis Functions Method

To make a brief description, we give restatement of the boundary value problems for elliptic partial differential equation of second order

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(P),\ P=(x,y)\in\Omega, \tag{1}$$

$$u(P)=\bar{u}(P),\ P=(x,y)\in\Gamma_D, \tag{2}$$

$$\frac{\partial u(P)}{\partial n}=\bar{q}(P),\ P=(x,y)\in\Gamma_N, \tag{3}$$

Where $\Omega\subset R^2$ is a 2D physical domain, $\bar{u}(P)$ and $\bar{q}(P)$ are the prescribed Dirichlet and Neumann boundary conditions, with $\Gamma_D\bigcup\Gamma_N=\partial\Omega$, $\Gamma_D\bigcap\Gamma_N=\phi$.

The basic theory of the conical radial basis function method is similar with the other RBF-based collocation method, i.e., the numerical solution of the BVP (1)-(3) can be derived by the following general formulation

$$\tilde{u}(P)=\sum_{j=1}^{M}\alpha_j\varphi\left(\left\|P-P_j\right\|_2\right), \tag{4}$$

Where M is the total number of source points $P_1, P_2, ..., P_M$ on the whole physical domain $\bar{\Omega} = \Omega \bigcup \partial\Omega$, and $\alpha_1, \alpha_2, ..., \alpha_M$ are the unknown coefficients, $\varphi\left(\left\|P - P_j\right\|_2\right) = r_j^m$ is the conical radial basis function， m is a positive odd integer in the conical radial basis function, r_j is the Euclidean norm distance between points $P = (x, y)$ and $P_j = (x_j, y_j)$.

We denote $P_1, P_2, ..., P_{M_I}$ the collocation points inside the domain Ω, $P_1, P_2, ..., P_{M_D}$ the collocation points on the Dirichlet boundary Γ_D and $P_1, P_2, ..., P_{M_N}$ the collocation points on the Neumann boundary Γ_N with the total collocation number $M = M_I + M_D + M_N$. The generation of collocation points are generated by the Chebyshev-type schemes in the following section [6].

3. The Chebyshev-Type Schemes

The key point of the new conical radial basis function method is to use the non-uniformly distributed Chebyshev-type schemes, which is generated in the interval. The computational cost remains the same as the traditional conical radial basis function method and there is no need to consider the fictitious points. The definite generations on each direction of the physical domains for three Chebyshev-type nodes are shown as below.

The Chebyshev-Gauss Scheme

$$P_j = \cos\frac{(2j+1)\pi}{2n+2}, 0 \le j \le n. \tag{5}$$

The Chebyshev-Gauss-Radau Scheme

$$P_0 = 1, P_j = \cos\frac{2\pi j}{2n+1}, 1 \le j \le n. \tag{6}$$

The Chebyshev-Gauss-Lobatto Scheme

$$P_j = \cos\frac{\pi j}{n}, 1 \le j \le n-1. \tag{7}$$

The main differences among these three newly-proposed schemes lie in the position of the collocation points. The figures of the three schemes will be shown in the following numerical section to verify the performance of the proposed schemes.

4. Numerical implementation

By forcing Equation (4) to satisfy Equations (1)-(3) at all collocation points, we have the following equations.

$$\sum_{i=1}^{M}\alpha_i\left[\frac{\partial^2\varphi(P_i, P_j)}{\partial x^2} + \frac{\partial^2\varphi(P_i, P_j)}{\partial y^2}\right] = f(P_i), P_j = (x_j, y_j) \in \Omega, j = 1, ..., M_I, \tag{8}$$

$$\sum_{i=1}^{M}\alpha_i\varphi(P_i, P_j) = \bar{u}(P_i), P_j = (x_j, y_j) \in \Gamma_D, j = M_I + 1, ..., M_I + M_D, \tag{9}$$

$$\sum_{i=1}^{M}\alpha_i\frac{\partial\phi(P_i, P_j)}{\partial n} = \bar{q}(P_i), P_j = (x_j, y_j) \in \Gamma_N, j = M_I + M_D + 1, ..., M. \tag{10}$$

This procedure is the same for the traditional source points and the source points generated by the Chebyshev-type schemes (5)-(7).

5. Measurement of Interpolation Matrix Conditioning

The L^2 condition number of a nonsingular square matrix A in Eqs. (8)-(10) is defined by matrix norm $\text{Cond}(A) = \|A\| \cdot \|A^{-1}\|$. The matrix can be decomposed by using singular value decomposition $A = UDV^T$ with diagonal matrix D including diagonal elements $\sigma_1, \sigma_2, ..., \sigma_N \geq 0$.

If matrix system is perturbed as $A(x + \Delta x) = b + \Delta b$, we can get $b = \sum_{i=1}^{N} \beta_i u_i$, where $u_i, i = 1, 2, ..., N$ are the elements of U. The effective condition number can be defined as [7].

$$ECN(A,b) = \frac{\|b\|}{\sigma_N \sqrt{\left(\frac{\beta_1}{\sigma_1}\right)^2 + \cdots + \left(\frac{\beta_N}{\sigma_N}\right)^2}}. \qquad (11)$$

6. Numerical Example

To seek for the influence of noise, the boundary conditions are considered by adding random number $u = \bar{u} + \delta, q = \bar{q} + \delta$. We use the random number generator to provide random numbers in [-1,1] and suppose $\delta = \varepsilon \times \text{Rand}$.

To measure the accuracy, we compute the relative average errors of the exact and approximate solutions. The exact solution of (1)-(3) is taken as $u(x, y) = -\frac{1}{2}x + y + \frac{1}{4}(x^2 - y^2)$. The physical domain is $\Omega = \Omega_2 - \Omega_1$ with outer boundary $\Omega_2 = [-2, 2] \times [-2, 2]$ and inner boundary $\Omega_1 = [-1,1] \times [-1,1]$ with Dirichlet boundary. Here, we take the Chebyshev-Gauss Scheme (CGS) as an example to illustrate the numerical results. The total collocation point number for the CGS is $M = 348$ with corresponding computational point number $N_t = 1028$.

It is seen from Table 1 that for different noise level δ, the ECN increases while the Relative average errors and traditional condition numbers remain at the same level. This phenomenon is different with the other numerical methods. This may partially suggest the stability of the conical radial basis function method under the Chebyshev-Gauss Scheme.

Table 1: Relative average errors (RAE), condition numbers, effective condition numbers for the Chebyshev-Gauss Scheme (CGS)

	RAE	Condition number	ECN
noise δ	344	361	348
0.5	2.60×10^{-8}	7.92×10^{13}	25.48
0.1	2.60×10^{-8}	7.92×10^{13}	96.70
0.05	2.60×10^{-8}	7.92×10^{13}	148.79
0.01	2.60×10^{-8}	7.92×10^{13}	153.04
0.005	2.60×10^{-8}	7.92×10^{13}	559.83
0.001	2.60×10^{-8}	7.92×10^{13}	5.35×10^{3}

7. Conclusions

In this paper, the parameter-free conical radial basis function method is investigated under three different Chebyshev-type Schemes. The effective condition number is considered to evaluate the interpolation system of the conical radial basis function collocation method. The numerical results show that the effective condition number is not a better choice to evaluate the interpolation system. This is different with the other numerical methods.

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Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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